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Investment under Uncertainty: Timing and Capacity Optimization

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Contents

CHAPTER 1

Introduction

This thesis contributes to the real options and industrial organization research by studying how volume flexibility influences firm's investment decision under uncertainty in both monopoly and duopoly setting. More specifically, it investigates how the firm's ability to adjust the production output according to demand fluctuations affects the firm's decision to enter the market. Volume flexibility enables the firms to produce within the constraint of installed production capacities as such to adapt to market demand uncertainty. This motivates the study about the influence of volume flexibility on investment behavior, not only in a monopoly setting, but also in a oligopolistic framework, where the incumbent firm invest strategically to deter or accommodate the entrant firm. Furthermore, this thesis also contributes to the welfare analysis of policy instruments. Due to the decentralization of public resources, private firms are allowed to invest in these resources. These firms' investment decisions are driven by profit maximization rather than social welfare maximization when resources are centralized. Profit maximizing decision generates externality in a not fully competitive market and leads to market failure. Thus, policy instrument is needed to align investment decisions of a profit maximizing firm and the welfare maximizing social planner. In uncertain economic environment, the firm with the investment opportunity is holding an “option". To capture this characteristic, the real option approach is applied to study the investment decisions under demand uncertainty, with volume flexibility and subsidy support being introduced separately.

The real options approach considers the firm's investment opportunity as real options. Due to the uncertainty in economic setting, the firm can postpone the investment and wait for more information about the future uncertainty. Once the firm invests, the firm exercises or kills the option to wait for new information. The basic real options approach is explained by Dixit and Pindyck (1994). Early real options literature studies the decision of investment timing for a given capacity size. However, when the firm makes investment decisions, it is not only the timing that is important but also the size of the investment. By investing with a large capacity, the firm takes a risk in case of uncertain demand. On the one hand, the revenue may be too low to defray the investment costs if ex post demand turns out to be too low. On the other hand, a large capacity yields a high revenue if the realized demand is high. Dangl (1999) and Bar-Ilan and Strange (1999) are among the first to include the decision of optimal investment capacity. The standard result is that the uncertainty makes the firm invest later and more. This is because a larger uncertainty makes it optimal to wait for further information and delay the investment for high demand ranges.

There are strategic capacity interactions between firms when making investment decisions. It is well known that a firm can gain a first mover advantage by committing to an action ahead of its rivals, see Várdy (2004). The first investor can deter an entrant through preemptive investment in plant and equipment. According to Lieberman and Montgomery (1988), the investment capacity of first mover serves as a commitment to maintain a high level of production output, which is a price cut threat to decrease entrant's profit. The first mover successfully deters the entrant in these models of Spence (1977), Dixit (1980), Gilbert and Harris (1981) and Curtis and Ware (1987). Tirole (1988) discusses the incumbent's capacity choices to deter, accommodate, and block the entry of an entrant in Stackelberg model with fixed entry costs. To further analyze the investment decisions for both timing and capacity in strategic interactions, the real options framework is used and extended to the duopoly setting. Huisman and Kort (2015) analyze the deterrence and accommodation strategies of the first investor where both firms can invest to enter the market. Overinvestment by the first investor not only decreases the investment size of the second investor, but also delays entry of its competitor to prolong the monopoly privilege. Huberts et al. (2015a) show that entry deterrence can also be achieved by the incumbent's early investment timing rather than overinvestment when the incumbent is already operating in the market. This is because the incumbent firm invests earlier and in a smaller amount compared to the situation without potential entry. Lavrutich et al. (2016) extend the duopoly model by considering the hidden competition of a third firm and find that due to hidden competition the follower is more eager to invest. So entry deterrence strategy is more costly for the leader. A more detailed comparison and description of the monopoly and duopoly models with capacity decisions can be found in Huberts et al. (2015b).

There are several extensions to the problem of investment under uncertainty using real options approach. One extension is to introduce volume flexibility. A real life example of volume flexibility is the automobile industry, where manufacturers can produce multiple types of cars on a single assembly line with the Flexible Manufacturing System, see Boonman (2014). When the market demand for one type of cars drops, the manufacturer can modify the assemble line to decrease the output of the less popular cars and increase the production of the more popular cars. Operations management literature studies the investment of volume flexibility with discrete time models, e.g., see Van Mieghem and Dada (1999), Goyal and Netessine (2007), Anupindi and Jiang (2008), Goyal and Netessine (2011). However, they cannot analyze the optimal investment timing from a continuous time perspective. Hagspiel et al. (2016) fill the gap by taking the real options approach and analyze a monopoly firm's investment decisions of timing and capacity under volume flexibility. This is similar to Dangl (1999), but Hagspiel et al. (2016) considers also the situation that market demand can be so large that the firm produces up to capacity right after investment.

Some literatures introduce policy instruments to motivate earlier investment from a real options perspective. One common instrument is to use price regulation such as the price cap to regulate the delayed investment under uncertainty. According to McDonald and Siegel (1986), an unregulated monopolist delays investment when there is demand uncertainty. This is because the firm cannot appropriate all benefits, but does incur all costs. So the monopolist tends to delay investment longer. If a regulator wants to correct this only by the price cap, Dobbs (2004) thinks the first-best outcome cannot be reached as one instrument is used for two goals: optimal investment ex-ante and optimal post-investment pricing. Building on Dobbs (2004), Evans and Guthrie (2012) introduce scale economics for capacity expansion where grouping investments across time is cost efficient, and show the price cap should be lowered. Willems and Zwart (2017) assume constant returns to scale in capacity expansion where it is not optimal to group investments. By assuming that the monopolist has private information on investment costs, Willems and Zwart (2017) find that the optimal mechanism can be implemented as a revenue tax that increases with the level of the price cap. For lumpy investment and cost information asymmetry, Broer and Zwart (2013) show that price cap should decrease as a function of the monopolist's chosen investment timing.

Market prices are generally influenced by the output quantity. For some industries, the regulator cares not only about the investment timing, but also the size of investment. So the policy instrument that regulates investment size is also used in practice. For instance, in agriculture and energy industry, investments are often subsidized by the government. The purpose of investment subsidies is to encourage private firms' investments to achieve some social objectives, like to increase the green energy consumption. The European Commission has set a binding target of 20% energy consumption from renewable sources by 2020 and at least 27% by 2030. However, an electricity producer might hesitate to invest in renewable technology due to high investment costs compared to the fossil fuels. So the energy market has less incentive to deliver the desired level of renewable consumption, which implies that support schemes should be provided to boost the investment activities in the renewable energy sector. Such support schemes take different forms such as R&D support, investment support[1], feed-in tariffs[2], quota[3], and green certificate[4] etc. A significant body

[1] Investment subsidies are to help overcome the barrier of a high initial investment. Investment subsidies are usually implemented by means of the fiscal system such as rebates on general energy taxes, lower VAT rates, tax exemption for green funds, etc.

[2] A regulatory, minimum guaranteed price per unit of produced electricity to be paid to the producer.

[3] A regulatory framework within which the market has to produce, sell or distribute a certain amount of energy from renewable sources.

[4] A tradable commodity proving that certain electricity is generated using renewable energy sources. Pur-

of real options literature focuses on the subsidized investment decisions under demand or policy uncertainty, see Pawlina and Kort (2005), Boomsma et al. (2012), Boomsma and Linnerud (2015), Adkins and Paxson (2015), and Chronopoulos et al. (2016). The common conclusions of these contributions are that subsidy support provides incentives for earlier investment. Furthermore, the opportunity to retract subsidy support in the future accelerates investment, whereas the possibility to introduce subsidy support in the future delays investment. Most of these literatures take the investment size as given and analyze the firm's decision on investment timing. The objective of the firms is to achieve profit maximization, which is different from the social welfare objective. This results in differences between investment decisions of a profit maximizer and a welfare maximizer, see Huisman and Kort (2015). The void of literature in optimal policy to align investment timing and investment size of market players with different objectives, i.e. a profit maximizer and a social welfare maximizer, is also an inspiration of this thesis.

This thesis addresses the above mentioned economic problems and analyzes the optimal investment decisions about timing and capacity for investment under uncertainty. There are three main chapters, where the lumpy investment under uncertainty is considered as continuous-time optimal stopping problem and analyzed from the real options perspective. In the continuation region the firm waits with investing and holds an option to invest, and in the stopping region it is optimal for firm to invest immediately.

Chapter 2 studies the investment timing and capacity decisions of a monopoly firm, where the firm has volume flexibility and can adjust the output level within the constraint of invested capacity. Hagspiel et al. (2016) analyze a market with unbounded size, whereas this chapter considers a market that is bounded. More specifically, compared with Hagspiel et al. (2016), this chapter considers a different demand function, which leads to different results. For instance, Hagspiel et al. (2016) conclude that the utilization rate, the proportion of capacity that is used for production right after investment, decreases with market uncertainty. Whereas this chapter shows that with a bounded market size, the utilization rate increases with market uncertainty in a fast growing market. The reason is that for a fast growing market that is bounded, the optimal investment capacity is already at high levels when the uncertainty is low. The increase in uncertainty does not significantly increase the optimal capacity, but delays the optimal investment timing a lot. Thus, the market demand is high when the firm invests, and a larger proportion of capacity is used for production. In addition, it shows that in a fast growing market the firm produces below capacity right after investment. If the market is slowly growing or shrinking, firm produces up to capacity right after investment. In the intermediate case, the firm produces up to capacity right after investment when uncertainty is low and below capacity when uncertainty is high.

Chapter 3 considers strategic capacity investment in a duopoly setting, where the first investor, the leader, always produces up to full capacity; and the second investor, the follower,

chasing a green certificate equals to purchasing a claim that the certificate owner consumes energy from the renewable portion of the whole energy in the electricity grid.

can choose volume flexibility, i.e., it can adjust output levels according to market demand. This chapter focuses on the influence of volume flexibility on the strategic interactions between firms. Thus, on one hand it extends Chapter 2 by introducing competition into the investment problems with volume flexibility. On the other hand, it is a generalization of Huisman and Kort (2015), where both firms always have to produce up to capacity. The results show that volume flexibility yields higher value for the follower. Compared with the situation of a nonflexible competitor, the leader has more incentive to accommodate rather than to deter the entry of its competitor. The reason is that the leader also benefits from its competitor's flexibility. More specifically, follower's volume flexibility affects the market price such that it does not fluctuate greatly and this is beneficial for both players. Moreover, the leader has a higher value compared with the follower. This implies that the leader's first mover advantage is not overcome by the follower's technological advantage in flexibility.

Chapter 4 investigates the optimal subsidy policy that aligns investment timing and capacity of a firm (profit maximizer) and social planner (welfare maximizer). Compared to the literatures on regulation, this chapter considers not only to regulate the investment timing, but also the investment size. Besides, the regulation literatures assume that the demand structure is non-linear and the monopolist invests later than the social planner. Apart from non-linear demand structure, this chapter also studies a linear demand structure and shows that the monopolist invests at the same time as the social planner. Furthermore, price subsidies and reimbursed investment cost are considered under linear and non-linear demand structures. The analytical results show that subsidy makes the firm invest earlier than without subsidy, which is consistent with the literature. However, under linear demand, the subsidy implemented from the beginning cannot align the firm's and social planner's investment decisions. This is because the monopoly firm invests less than the social optimal capacity. We show that there exists a conditional subsidy that aligns investment decisions. The conditional subsidy requires to introduce subsidy when the socially optimal investment is triggered. Then with the appropriate subsidy rates, the monopoly firm can be motivated to invest with the socially optimal capacity size. Under non-linear demand, it is possible to align the decisions by either implementing the investment subsidy from the beginning, or introducing the subsidy at the socially optimal investment timing. When the two decisions are aligned, the subsidy maximizes social welfare.

CHAPTER 2

Volume Flexibility and Capacity Investment

This chapter considers the investment decision of a firm where it has to decide about the timing and capacity. On the one hand, we obtain that in a fast growing market, the firm produces below capacity right after investment. The utilization rate (the proportion of capacity that is used for production right after the investment) increases with market uncertainty for a very big market trend, and shows no monotonicity for a moderately large market trend. On the other hand we get that, for a slowly growing or shrinking market, the firm produces up to capacity right after investment. In the intermediate case, the firm produces up to capacity right after investment when uncertainty is low and below capacity when uncertainty is high. The utilization rate in this case decreases with the market uncertainty. This chapter is based on Wen et al. (2017).

2.1 Introduction

When entering a market, it is not only the timing that is important, but also the size of the production capacity with which the firm enters. By investing in a large capacity, the firm faces large investment cost, but can generate a high revenue in periods of high demand on the one hand. On the other hand, if a firm is dedicated to producing at full capacity, it may face a decline in revenues in case of a low demand realization. In this model we allow for volume flexibility. It is defined as the firm can operate profitably at different output levels according to Sethi and Sethi (1990). This enables the firm to produce less when demand is low, and keep part of the invested capacity idle. In this way, volume production reduces the downside risk that a firm takes.

Most of the literatures that study investment decision from real options perspective focus on the optimal investment timing, taking the size of the investment as given (see Dixit and Pindyck (1994); Trigeorgis (1996) for an overview). In this chapter, we determine not only the optimal timing but also the optimal capacity size. Several contributions show that if a monopolist is allowed to choose the size of its investment, it invests later with larger capacity for higher market uncertainty (see Manne (1961); Bar-Ilan and Strange (1999)). In a duopoly setting, there are strategic capacity interactions between firms. The first investor can choose to overinvest in order to decrease the investment size of its competitor on the one hand, and delay the entry of its competitor to prolong the monopoly privilege on the other hand (see Huisman and Kort (2015)).

Among the early contributions that consider flexibility is the static model by Van Mieghem and Dada (1999). They look at the effect of postponement in capacity, output and price decisions to the moment that uncertainty is resolved. Compared with production postponement, the price postponement makes the investment decision relatively insensitive to uncertainty. Chod and Rudi (2005) consider a firm that can use one flexible resource to produce two goods in a two-stage model. The optimal capacity of flexible resource is found to be always increasing in both demand variability and demand correlation. In a three-stage model, Anupindi and Jiang (2008) consider a situation when production can be decided before or after the demand realization, but the capacity decisions are made ex ante and pricing decisions ex post. They find that in a more volatile market firms invest with a larger capacity. By discretizing the dynamic of demand through binomial lattice, Fontes (2008) compares a fixed capacity strategy with a flexible capacity strategy and finds that an increase in flexibility leads to a higher predicted value of the project. In continuous time models, Brennan and Schwartz (1985), McDonald and Siegel (1985), and Adkins and Paxson (2012) consider the possibility to switch from operation to suspension and back to operation at a certain cost. In this chapter we investigate the flexibility to adjust production between zero and the invested capacity level at any time.

This chapter is closely related to Dangl (1999) and Hagspiel et al. (2016). Dangl (1999) however, does not take into account the possibility that the market demand is so high that the firm produces up to capacity right after the investment, whereas Hagspiel et al. (2016) take that into consideration and conclude that the utilization rate decreases when the market uncertainty increases. Compared to Hagspiel et al. (2016), we adopt a slightly different demand function, which, however, leads to new implications. The difference in demand function is that the market size is unbounded in the work of Hagspiel et al. (2016). The demand function used in this chapter implies a bounded market size. Market size is related to the number of potential customers or sellers of a product or service. Consider for instance the market of agricultural machines like the harvesters in a region like the Netherlands. The population of farmers and the area of farmlands are limited. This results in an upper bound on demand.

Our main results are the following. First, we find that under a very large market trend, right after the investment the utilization rate increases with uncertainty. This is due to the fact that when market trend is very large and uncertainty is low, the firm invests in a capacity relatively close to the maximal size of the market. Higher uncertainty makes it optimal to invest later, i.e. when demand is larger. As a response to large demand, the firm is willing to produce more, while at the same time the capacity increases. This increase is however relatively small because it was already large. Consequently, it turns out that production increases more than capacity does with uncertainty. This leads to the counterintuitive result that the utilization rate increases with uncertainty. However, an intermediate market trend still results in an utilization rate that decreases with uncertainty as in Hagspiel et al. (2016). A moderately large market trend in turn yields a non-monotonic utilization rate.

We also find that, when the market trend is large, the firm does not produce up to capacity right after the investment; when the market trend is small, the firm produces up to capacity; when the market trend is intermediate, there exists a threshold uncertainty level such that the firm produces below capacity right after the investment above this threshold and produces up to capacity below this threshold. Lastly, we find interesting results related to the effect of market trend on investment timing: The optimal timing is delayed for a larger trend in a less volatile environment and accelerated in a more volatile environment. This results from the large capacity installment for a small market trend under high market uncertainty. As the market grows faster, the capacity does not increase a lot due to the bounded market size we imposed. When the market trend increases, the firm then actually prefers to invest earlier.

The rest of this chapter is structured as follows. Section 2.2 describes the monopoly investment problem. The optimal investment decision is determined and analyzed in Section 2.3. A numerical analysis is provided in Section 2.4. Section 2.5 concludes.

2.2 Model Setup

Consider a monopolist that is considering to undertake an investment to enter a market with uncertain demand. The market price at any time $t \geq 0$, is given by

$$p(t) = X(t)(1 - \gamma q(t)),$$

with $q(t)$ being the firm's output and $\gamma > 0$ a constant. Note that in this inverse demand function, the market is bounded above in such a way that $q(t) \leq 1/\gamma$ holds[1]. Demand uncertainty is modeled by $\{X(t) | t \geq 0\}$ following the geometric Brownian motion

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW_t,$$

where $X(0) > 0$, α is the trend parameter, σ ($\sigma > 0$) is the volatility parameter, and dW_t is the increment of a Wiener process. The firm is risk-neutral and the discount rate $r > 0$ is assumed to satisfy $r > \alpha$ and $r > \sigma^2 - \alpha$. The first inequality is standard and the problem

[1]There is no upper bound for the demand function $p(t) = X(t) - \gamma q(t)$ adopted in Hagspiel et al. (2016).

makes sense only when it holds. If this inequality does not hold, by choosing to invest later, the integral representing the discounted revenue flow could be made infinitely larger. Thus it is always better for the firm to delay the investment, and the optimum would not exist (see Dixit and Pindyck (1994)). The second inequality is because the Brownian motion process $\{1/X(t)|t \geq 0\}$ has a trend of $\sigma^2 - \alpha$, which should also be smaller than r to avoid delaying investment forever for the same argument[2]. From now on, we drop the argument of time whenever there can be no misunderstanding.

Once the investment is made, the firm becomes active and can decide on the production level, which is bounded from above by the installed capacity $K \geq 0$. The unit cost for acquiring capacity is $\delta > 0$, and the unit cost for production is $c > 0$.

2.3 Optimal Investment Decision

This section is about the optimal investment decision of a monopoly firm. We first determine the firm's optimal production decisions and corresponding instantaneous profit $\pi(X,K)$ for a given K. Once the firm becomes active in the market with installed capacity $K \geq 0$, it chooses at level X of an output to maximize the profit flow, i.e.

$$\pi(X,K) = \max_{0 \leq q \leq K} (p-c)\,q = \max_{0 \leq q \leq K} [X(1-\gamma q) - c]\,q. \tag{2.1}$$

There are three possibilities for the firm's output levels. Production will be temporarily suspended when X falls below c and resumed when X rises above c. For the resumed production, the firm either produces below capacity or up to capacity. The optimal production and corresponding profit are determined in the following proposition.

Proposition 2.1. *For invested capacity $K \geq 0$, and level $X > 0$, the optimal monopoly production output is*

$$q^*(X,K) = \begin{cases} 0 & 0 < X < c, \\ \frac{X-c}{2\gamma X} & X \geq c \text{ and } K > \frac{X-c}{2\gamma X}, \\ K & X \geq c \text{ and } 0 \leq K \leq \frac{X-c}{2\gamma X}. \end{cases} \tag{2.2}$$

[2]It should be noted that the optimal output, corresponding profit and option value of the firm are proportional to $1/X$ when the firm produces below capacity right after the investment, as can be inferred from (2.2), (2.3) and (2.8) later on.

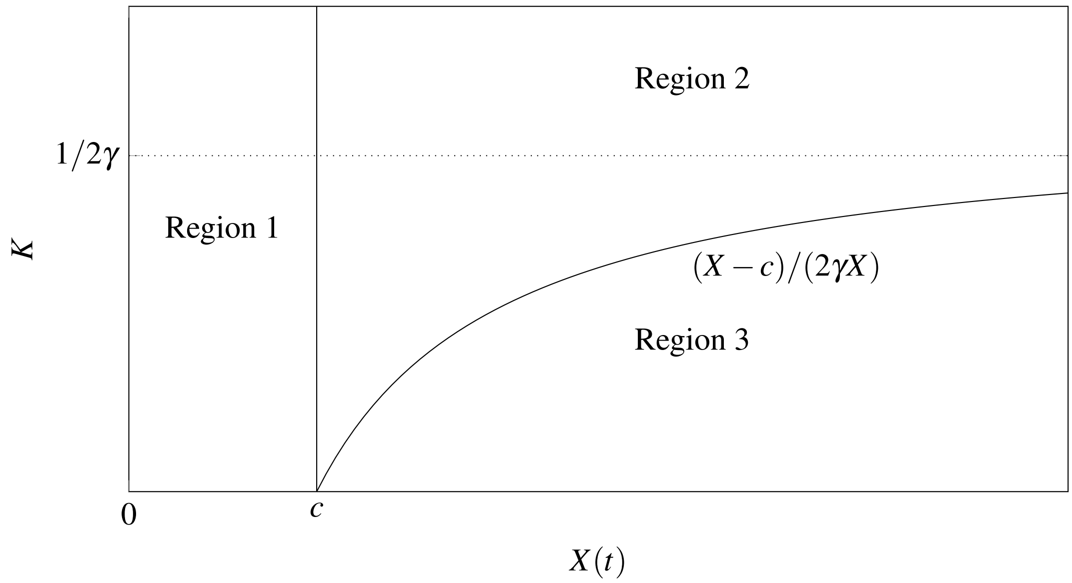


Figure 2.1: Comparison of investment capacity K and optimal production outputs $q(X,K)$. In Region 1, $q(X,K) = 0$; in Region 2, $q(X,K) = (X-c)/(2\gamma X)$; and in Region 3, $q(X,K) = K$.

The corresponding profit is

$$\pi(X,K,q^*) = \begin{cases} 0 & 0 < X < c, \\ \frac{(X-c)^2}{4\gamma X} & X \geq c \text{ and } K > \frac{X-c}{2\gamma X}, \\ [X(1-\gamma K) - c]K & X \geq c \text{ and } 0 \leq K \leq \frac{X-c}{2\gamma X}. \end{cases} \tag{2.3}$$

The comparison between production output and investment capacity is illustrated in Figure 2.1, where the line $X = c$ and the curve $(X-c)/(2\gamma X)$ divide the (X,K)-space into three regions. In Region 1, where $0 < X < c$, there is no production. Region 2 is to the right of $X = c$ and above the curve $(X-c)/(2\gamma X)$. It is the region where the optimal output level is lower than the invested capacity. Region 3 is below the curve $(X-c)/(2\gamma X)$, where the production is constrained by the capacity and the firm produces an output level being equal to the installed capacity.

The firm solves an optimal stopping problem, and can be formalized as

$$\sup_{T\geq 0, K\geq 0} E\left[\int_T^\infty \pi(X(t),K)\exp(-rt)dt - \delta K \exp(-rT) \middle| X(0) = X\right], \tag{2.4}$$

which is conditional on the available information at time 0 with $X(0)$ set equal to X, where T is the moment of investment and $\pi(X,K)$ is the maximum profit of the firm at time $t \geq T$ if capacity K has been invested.

Let X^* be the value of the Brownian motion where the firm is indifferent between continuation and stopping, and let the corresponding acquired capacity be K^*. For $X(0) > X^*$, the firm is in the stopping region and it is optimal to invest immediately. For $0 < X(0) < X^*$, demand is too low to undertake investment. Then the firm is in the continuation region and waits with investing until X reaches X^*. We study the scenario that $X(0) < X^*$. So it is not optimal to invest at the initial point of time. The optimal investment time T equals to the first time that the stochastic process X that starts at $X(0)$ at time zero reaches X^*. Denote by $V(X,K)$ the value after investment given that the level of the geometric Brownian motion is X and capacity K has been installed. Next we obtain a dynamic programming equation. We start by applying Ito's lemma to $V(X,K)$ (see, e.g., Dixit and Pindyck (1994))

$$dV = \frac{\partial V}{\partial X}\alpha X dt + \frac{\partial^2 V}{\partial X^2}\frac{1}{2}\sigma^2 X^2 dt + \frac{\partial V}{\partial X}\sigma X dW, \tag{2.5}$$

where, for the sake of simplicity, we have omitted the arguments of the function $V(X(t),K)$. Then it follows that V satisfies the Bellman equation

$$rV = \pi + \frac{1}{dt}E[dV]. \tag{2.6}$$

Substitution of (2.5) into the Bellman equation and also using the fact that $E[\frac{\partial V}{\partial X}\sigma X dW] = 0$ result in the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V(X,K)}{\partial X^2} + \alpha X \frac{\partial V(X,K)}{\partial X} - rV(X,K) + \pi(X,K) = 0. \tag{2.7}$$

Substitution of (2.3) into (2.7), and employing value matching and smooth pasting give the value after the investment:

$$V(X,K) = \begin{cases} L(K)X^{\beta_1} & 0 < X < c, \\ M_1(K)X^{\beta_1} + M_2 X^{\beta_2} & \\ +\frac{1}{4\gamma}\left[\frac{X}{r-\alpha} - \frac{2c}{r} + \frac{c^2}{(r+\alpha-\sigma^2)X}\right] & X \geq c \text{ and } K > \frac{X-c}{2\gamma X}, \\ N(K)X^{\beta_2} + \frac{(1-\gamma K)K}{r-\alpha}X - \frac{cK}{r} & X \geq c \text{ and } 0 \leq K \leq \frac{X-c}{2\gamma X}, \end{cases} \tag{2.8}$$

in which

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{2.9}$$

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < -1. \tag{2.10}$$

The expression and derivation of $L(K)$, $M_1(K)$, M_2 and $N(K)$, the proofs of their signs and the following propositions are presented in the Appendix. $L(K)X^{\beta_1}$ is positive and increases with X. The monopolist does not produce right after investment because the demand is too small. $L(K)X^{\beta_1}$ represents the option value to start producing in the future and this happens as soon as X reaches c. $M_1(K)X^{\beta_1}$ is negative and corrects for the fact that if X reaches $c/(1-2\gamma K)$, the firm's output will be constrained by the installed capacity size K. $M_2(K)X^{\beta_2}$ is also negative and corrects for the positive quadratic form of cash flows such that if X drops below c, the monopolist would temporarily suspend the production. $N(K)X^{\beta_2}$ is positive and stands for the option value that if X falls below $c/(1-2\gamma K)$, the firm would produce below capacity.

We find the optimal investment decision in two steps. First, for any given level of the geometric Brownian motion X, the optimal value of K is found by maximizing $V(X,K) - \delta K$. Second, let the value before investment be AX^{β}. Then the optimal investment threshold and capacity level are derived. The two steps are summarized in the following proposition, where $\bar{\sigma} > 0$ is a value of the drift parameter that determines if the firm produces below or up to capacity right after investment for certain trend parameter values[3]. $\bar{\sigma}$ is such that

$$\bar{\sigma}^2 = \frac{4\sqrt{r\Lambda(\Lambda-\alpha^2)(r-\alpha)} - 2\left(\Lambda-\alpha^2\right)(2r-\alpha)}{\Lambda-(2r-\alpha)^2}, \tag{2.11}$$

with $\Lambda = \left(\frac{2\delta r(r-\alpha)-\alpha c}{c}\right)^2$.

Proposition 2.2. *There are two possibilities regarding the firm's investment decision:*

1. *Suppose either $\alpha > \delta r^2/(c+\delta r)$, or both $r - c/\delta < \alpha \leq \delta r^2/(c+\delta r)$ and $\sigma > \bar{\sigma}$. Then the firm does not produce up to capacity right after the investment. For any $X \geq c$, the optimal value of K that maximizes $V(X,K) - \delta K$ is*

$$K(X) = \frac{1}{2\gamma}\left[1 - \frac{c}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right], \tag{2.12}$$

and the optimal investment threshold X^* is implicitly determined by

$$\frac{\beta_1-\beta_2}{\beta_1}M_2X^{\beta_2} + \frac{1}{4\gamma}\left[\frac{X(\beta_1-1)}{\beta_1(r-\alpha)} - \frac{2c}{r} + \frac{c^2(\beta_1+1)}{\beta_1(r+\alpha-\sigma^2)X}\right] - \delta K(X) = 0. \tag{2.13}$$

[3] $\bar{\sigma}$ is only defined for the situation of $r - c/\delta < \alpha < \delta r^2/(c+\delta r)$.

If $X(0) < X^*$, the optimal capacity is $K^* = K(X^*)$. If $X(0) \geq X^*$, the firm invests in capacity $K^* = K(X(0))$ immediately at $t = 0$.

2. *Suppose either $\alpha \leq r - c/\delta$, or both $r - c/\delta < \alpha \leq \delta r^2/(c + \delta r)$ and $\sigma \leq \bar{\sigma}$. Then the firm produces up to capacity right after the investment. For any $X \geq c$, the optimal value of K that maximizes $V(X,K) - \delta K$, $K(X)$, satisfies*

$$\frac{(\beta_2+1)\,F(\beta_1)}{2(\beta_1-\beta_2)}\frac{(1-2\gamma K)^{\beta_2}}{c^{\beta_2-1}}X^{\beta_2} + \frac{1-2\gamma K}{r-\alpha}X - \frac{c}{r} - \delta = 0, \tag{2.14}$$

and the optimal investment threshold X^* is implicitly determined by

$$\frac{\beta_1-\beta_2}{\beta_1}N(K)\,X^{\beta_2} + \frac{\beta_1-1}{\beta_1}\frac{(1-\gamma K)\,KX}{r-\alpha} - \frac{cK}{r} - \delta K = 0, \tag{2.15}$$

with $K = K(X)$. If $X(0) < X^*$, the firm invests in capacity $K^* = K(X^*)$. If $X(0) \geq X^*$, the firm invests in capacity $K^* = K(X(0))$ immediately at $t = 0$.

Besides presenting the optimal investment threshold and capacity level, Proposition 2.2 also shows how the market affects the flexible firm's production decision right after the investment. This is illustrated in Figure 2.2. If the market is growing fast ($\alpha > \delta r^2/(c + \delta r)$), right after the investment the firm chooses an output below the installed capacity. The initially unused capacity can be employed later to meet an increased market demand. However, if the market is growing very slowly or even shrinking ($\alpha \leq r - c/\delta$), the firm produces at full capacity right after investment[4]. If the market trend is at an intermediate level ($r - c/\delta < \alpha \leq \delta r^2/(c + \delta r)$), the market uncertainty plays a decisive role in the decision of whether to produce up to full capacity right after the investment. In a more volatile environment ($\sigma > \bar{\sigma}$), producing below capacity makes the extra capacity idle. The extra capacity will be used when the price level is higher. For a more certain environment ($\sigma \leq \bar{\sigma}$), such an extra capacity is not needed, and the firm produces up to full capacity right after the investment.

2.4 Numerical Analysis

This section focuses on the influence of the market trend and uncertainty on the investment decision and the utilization rate right after the investment. The utilization rate is equal to the ratio q^*/K^* with $q^* = q(X^*, K^*)$. It gives insight into the overall slack of the firm right after investment. The capacity utilization tends to fluctuate with business cycles as the firm adjusts output levels in response to changing demand. Low capacity utilization is a concern for the authority and the firm because it means a large amount of the installed

[4]Note that we allow for $r < c/\delta$.

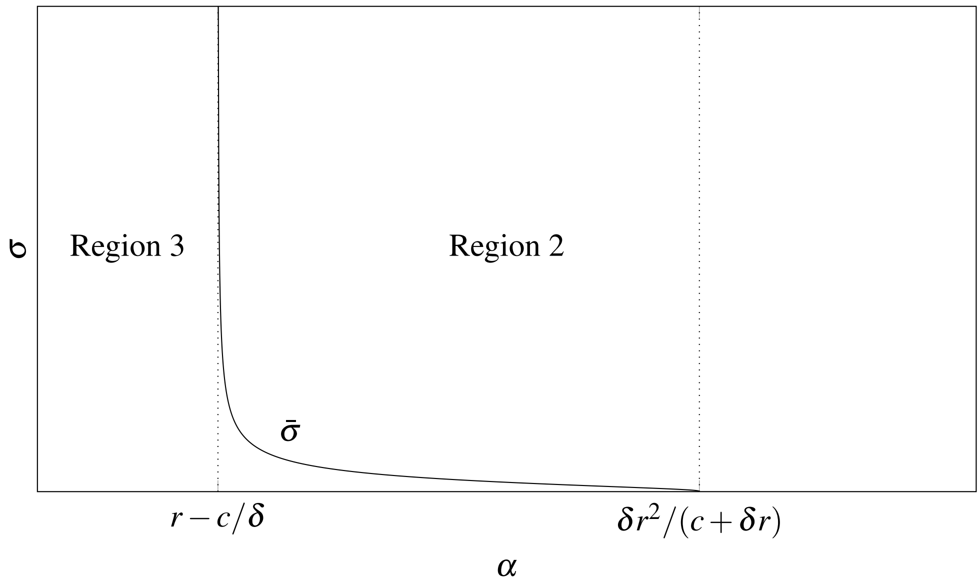


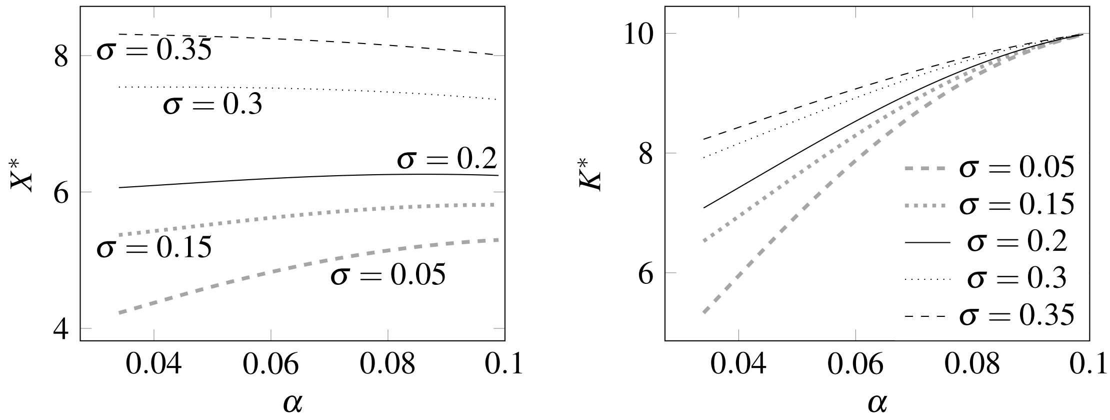
Figure 2.2: Illustration of market trend α and uncertainty σ affecting the firm to produce below capacity (Region 2), or up to capacity (Region 3), right after the investment.

capacity is idle and stimulative efforts are needed to increase the market demand. It has been shown that for the unbounded demand function $p(t) = X(t) - \gamma q(t)$, at given level of the market trend, the utilization rate decreases significantly with market uncertainty (see Hagspiel et al. (2016)). However, this section shows that for our model, if the market trend is large enough, the utilization rate increases with market uncertainty.

2.4.1 Market trend

We first look at how market trend affects the optimal investment timing and capacity when the firm produces below capacity right after the investment. As shown in Figure 2.3 we have that, when the market uncertainty is low, both the optimal investment time and the investment capacity increase with market trend. This is because when deciding how much to invest in a less volatile environment, the firm considers the market increase after the investment and installs a large capacity in case of a high market demand, which makes it reasonable to invest later. However, when the market uncertainty is high, Figure 2.3a shows that the firm invests slightly earlier for a larger market trend. The reason is that in a highly volatile environment, the firm still invests in a larger capacity for a larger market trend. But since the capacity level is already at a high level when the market grows slowly (Figure 2.3b), with a larger market trend the capacity does not increase a lot, and the resulting effect on investment timing is low.[5] This makes that in a higher uncertain environment, the firm prefers to invest earlier when the market trend goes up, because the firm is more eager to invest in such a market.

[5] Note that with this demand function, the optimal capacity size is always below $1/(2\gamma)$.



(a) Investment timing

(b) Investment capacity

Figure 2.3: Illustration of investment timing and capacity as function of market trend α under different uncertainty levels σ when producing below capacity right after the investment. Parameter values are $r = 0.1$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

The influence of the market trend α on the utilization rate q^*/K^* is shown in Figure 2.4. Regardless of the uncertainty level, the utilization rate decreases with α. This is because when deciding on capacity, the future market is considered. This implies that for a larger α, a larger capacity will be installed. At the same time, only the current market is important when deciding about the production amount. The current market size is small compared to the future market size when α is large. This makes that the production level is low compared to capacity, hence a low utilization rate results. Moreover, the utilization rate decreases less fast with α for larger σ. The intuition is that the rate of increase in the installed capacity is lower than that in production output for larger σ, since, as before, the optimal capacity is already close to its upper bound $1/(2\gamma)$ when σ is large.

2.4.2 Market uncertainty

When the market trend α is small, the utilization rate equals to 1 and is unaffected by the market uncertainty. When α is at an intermediate level, Figure 2.5 shows that the utilization rate is 1 for small market uncertainty σ, as is also illustrated in Figure 2.2, and decreases as market uncertainty σ increases. When α is large, the utilization rate increases with σ, and when α is moderately large, the utilization rate can both increase and decrease with σ. The intuition behind this is as follows.

If α is at intermediate level, i.e. $r - c/\delta < \alpha \leq \delta r/(c + \delta r)$, then the firm invests later in a larger capacity when σ goes up. This is shown in Figure 2.6. Given the other parameter values (r, γ, c, and δ) in Figure 2.6, the firm produces right after the investment below capacity for $\alpha = 0.02, \sigma > 0.2866$ and $\alpha = 0.03, \sigma > 0.1473$; and up to capacity for $\alpha = 0.02, \sigma \leq 0.2866$ and $\alpha = 0.03, \sigma \leq 0.1473$. The firm invests in a larger capacity in a more volatile environment, because more future uncertainty makes excess capacity

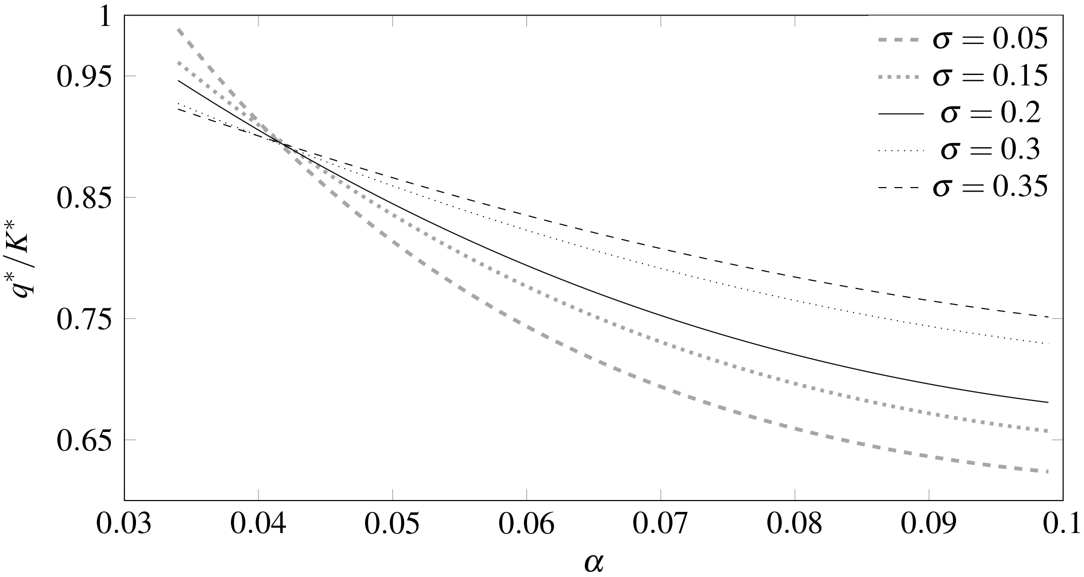


Figure 2.4: Illustration of utilization rate as function of market trend α under different uncertainty levels σ when producing below capacity right after the investment. Parameter values are $r = 0.1$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

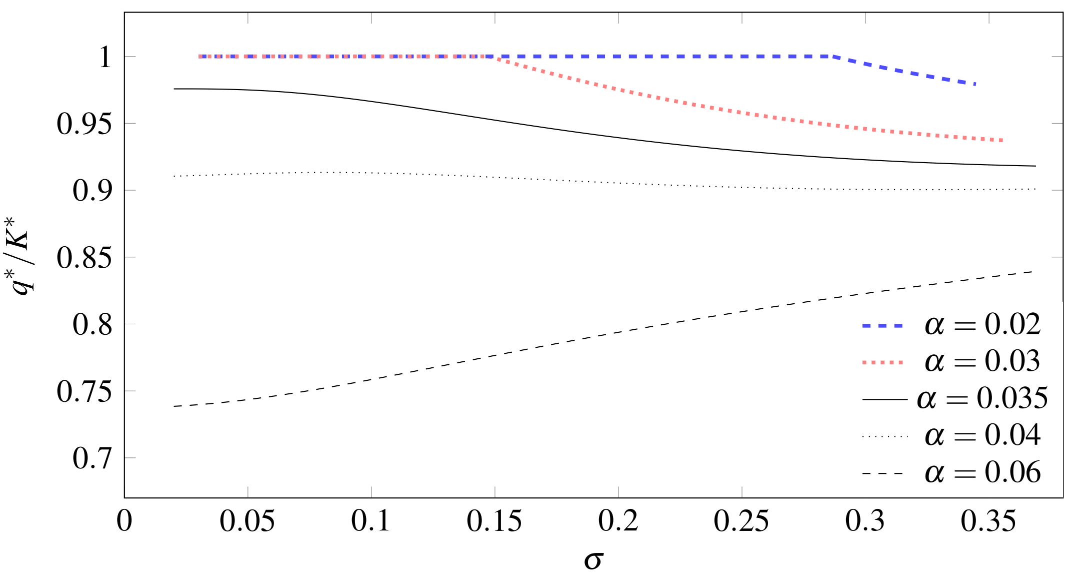
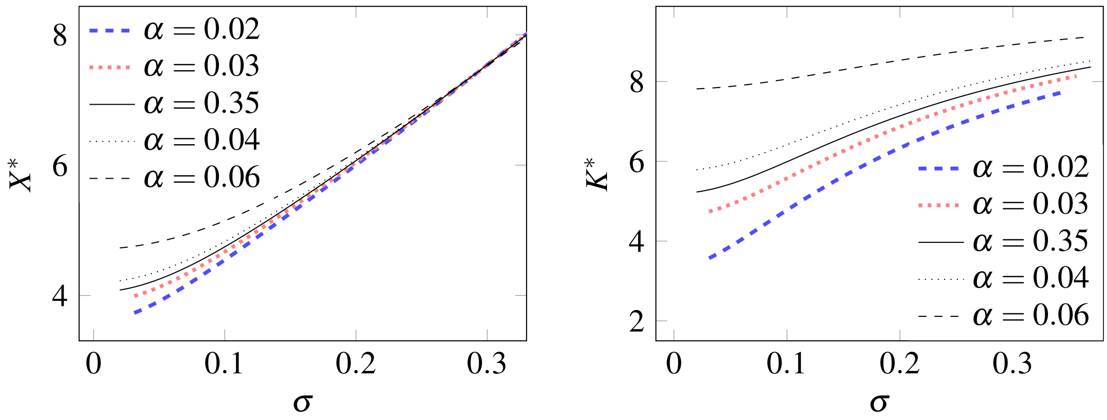


Figure 2.5: Illustration of utilization rate as function of the market uncertainty level σ under different market trends α. Parameter values are $r = 0.1$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.



(a) Investment timing

(b) Investment capacity

Figure 2.6: Illustration of investment timing and capacity as functions of uncertainty level σ under different market trends α. Parameter values are $r = 0.1$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

desirable to match upward demand shocks. Since the required capacity at the moment of investment is larger, so is the investment cost, the firm requests a higher output price when it invests, implying that the optimal threshold increases also. The delayed timing suggests that the output level is also increasing. But for an intermediate α, this happens in a more gradual way. Thus, the utilization rate decreases with uncertainty when the firm produces below capacity right after the investment. The finding that the utilization rate is decreasing with uncertainty for an intermediate α (take $\alpha = 0.02$ for example) is consistent with the findings in Hagspiel et al. (2016).

If α is large, i.e. $\alpha > \delta r^2/(c + \delta r)$, there are two possibilities. When α is very large, take $\alpha = 0.06$ for example, in Figure 2.6b, the optimal capacity is already at a high level for small σ. Then the capacity upper bound of $1/(2\gamma)$ is relatively close, so the capacity increases slowly with σ. However, the optimal investment timing is delayed a lot compared with the optimal capacity. This implies the output right after the investment increases quite a lot. Thus, for a very large α, the utilization rate increases with uncertainty. This result is not present in the work of Hagspiel et al. (2016), because because our model is based on different demand functions. In their work the market is not bounded, whereas in this chapter, we have that a positive market price requires the quantity to be always below $1/\gamma$. When α is moderately large, for example, $\alpha = 0.035$ or 0.04 in Figure 2.5, the utilization rate does not change monotonically with uncertainty. In fact, the opposite effects for intermediate α and very large α above occur here, causing the non-monotonicity.

2.5 Conclusion

This chapter analyzes the investment decisions of a monopoly firm with access to volume flexibility in a dynamic uncertain environment. In such an environment, not only the uncertainty, but also the market trend has significant qualitative effects on the timing, the investment capacity size, and the decision whether to produce up to capacity right after the investment. We show that a large (small) market trend corresponds to producing below (up to) capacity right after the investment. An intermediate market trend and an uncertainty level above (below) a certain threshold yields an output level below (up to) capacity right after the investment. The utilization rate is increasing with market uncertainty when the trend is very large, shows no monotonicity when the trend is moderately large, and decreases with uncertainty when the trend is intermediate. Moreover, we find that capacity increases and the utilization rate decreases with the market trend. However, the investment timing is delayed in a more certain market, but accelerated in a more volatile market.

A limitation of the model is that the firm can only invest once. If the firm can undertake several investments during its life time, then the decision to produce up to/below capacity after investment is probably going to be affected by the frequency and moments of investments, which could be an interesting topic for future research. Another interesting topic is to introduce competition by studying a duopoly framework. Then Huisman and Kort (2015), where firms are obliged to produce up to capacity, is extended by allowing the firms to produce below capacity. The implication is that the firm can no longer commit to a high production level, which leads to a significant change in the resulting strategic interactions.

2.6 Appendix

Proof of Proposition 2.1 Optimal output $q(X,K)$ equals to 0 when there is no production right after investment and equals to K when the firm produces up to capacity right after investment. When the firm produces below capacity right after investment, the optimal output $q^*(X,K)$ maximizes $[X(1-\gamma a)-c]q$. Substituting $q^*(X,K)$ into $[X(1-\gamma a)-c]q$ yields the corresponding profits.

Identification of $L(K)$, $M_1(K)$, M_2, and $N(K)$ Given the value function in different regions and according to the value matching and smooth pasting conditions at $X_1 = c$ and $X_2 = \frac{c}{1-2\gamma K}$, it holds that

$$\begin{aligned} L(K)X_1^{\beta_1} =& M_1(K)X_1^{\beta_1} + M_2X_1^{\beta_2} \\ &+ \frac{X_1}{4\gamma(r-\alpha)} - \frac{2c}{4\gamma r} + \frac{c^2}{4\gamma(r+\alpha-\sigma^2)X_1}, \end{aligned} \tag{2.16}$$

$$L(K)\beta_1X_1^{\beta_1-1} = \beta_1M_1(K)X_1^{\beta_1-1} + \beta_2M_2X_1^{\beta_2-1}$$

$$+\frac{1}{4\gamma(r-\alpha)}-\frac{c^2}{4\gamma(r+\alpha-\sigma^2)X_1^2}, \tag{2.17}$$

$$\begin{aligned} M_1(K)X_2^{\beta_1}+M_2X_2^{\beta_2}+\frac{X_2}{4\gamma(r-\alpha)}-\frac{2c}{4\gamma r}+\frac{c^2}{4\gamma(r+\alpha-\sigma^2)X_2} \\ =N(K)X_2^{\beta_2}+\frac{(1-\gamma K)K}{r-\alpha}X_2-\frac{cK}{r}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \beta_1M_1(K)X_2^{\beta_1-1}+\beta_2M_2X_2^{\beta_2-1}+\frac{1}{4\gamma(r-\alpha)}-\frac{c^2}{4\gamma(r+\alpha-\sigma^2)X_2^2} \\ =\beta_2N(K)X_2^{\beta_2-1}+\frac{(1-\gamma K)K}{r-\alpha}. \end{aligned} \tag{2.19}$$

Take

$$\begin{aligned} F(\beta)=&\frac{2\beta}{r}-\frac{\beta-1}{r-\alpha}-\frac{\beta+1}{r+\alpha-\sigma^2} \\ =&\frac{\beta\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)+r\left(2\alpha-\sigma^2\right)}{r(r-\alpha)(r+\alpha-\sigma^2)}. \end{aligned} \tag{2.20}$$

From (2.16) and (2.17), we get

$$\begin{aligned} M_2=&\frac{X_1^{-\beta_2}}{4\gamma(\beta_1-\beta_2)}\left[\frac{2c\beta_1}{r}-\frac{X_1(\beta_1-1)}{r-\alpha}-\frac{c^2(\beta_1+1)}{(r+\alpha-\sigma^2)X_1}\right] \\ =&\frac{c^{1-\beta_2}}{4\gamma(\beta_1-\beta_2)}\left(\frac{2\beta_1}{r}-\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1+1}{r+\alpha-\sigma^2}\right) \\ =&\frac{c^{1-\beta_2}}{4\gamma(\beta_1-\beta_2)}F(\beta_1). \end{aligned} \tag{2.21}$$

$M_1(K)$ can be derived from (2.18) and (2.19) as

$$\begin{aligned} M_1(K)=&\frac{X_2^{-\beta_1}}{\beta_1-\beta_2}\left\{\frac{1}{4\gamma}\left[\frac{X_2(\beta_2-1)}{r-\alpha}+\frac{c^2(\beta_2+1)}{(r+\alpha-\sigma^2)X_2}-\frac{2c\beta_2}{r}\right]\right. \\ &\left.+\frac{cK\beta_2}{r}+\frac{(1-\beta_2)(1-\gamma K)KX_2}{r-\alpha}\right\} \end{aligned}$$

$$
\begin{aligned}
=&\frac{X_2^{-\beta_1}}{\beta_1-\beta_2}\left[\frac{X_2(\beta_2-1)(1-2\gamma K)^2}{4\gamma(r-\alpha)}+\frac{c^2(\beta_2+1)}{4\gamma X_2(r+\alpha-\sigma^2)}\right.\\
&\left.-\frac{2c\beta_2(1-2\gamma K)}{4\gamma r}\right]\\
=&-\frac{X_2^{-\beta_1}c(1-2\gamma K)}{4\gamma(\beta_1-\beta_2)}\left(\frac{2\beta_2}{r}-\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2+1}{r+\alpha-\sigma^2}\right)\\
=&-\frac{c^{1-\beta_1}(1-2\gamma K)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)}F(\beta_2).
\end{aligned}
\tag{2.22}
$$

We get $L(K)$ from (2.16) as

$$
\begin{aligned}
L(K)=&M_1(K)+M_2X_1^{\beta_2-\beta_1}+\frac{X_1^{-\beta_1}}{4\gamma}\left[\frac{X_1}{r-\alpha}-\frac{2c}{r}+\frac{c^2}{(r+\alpha-\sigma^2)X_1}\right]\\
=&M_1(K)+\frac{X_1^{-\beta_1}}{4\gamma}\left[\frac{X_1}{r-\alpha}-\frac{2c}{r}+\frac{c^2}{(r+\alpha-\sigma^2)X_1}\right]\\
&+\frac{X_1^{-\beta_1}}{4\gamma}\left[\frac{2c\beta_1}{r(\beta_1-\beta_2)}-\frac{X_1(\beta_1-1)}{(r-\alpha)(\beta_1-\beta_2)}-\frac{c^2(\beta_1+1)}{(r+\alpha-\sigma^2)X_1(\beta_1-\beta_2)}\right]\\
=&M_1(K)+\frac{X_1^{-\beta_1}}{4\gamma}\left[\frac{X_1}{r-\alpha}\frac{1-\beta_2}{\beta_1-\beta_2}+\frac{2c}{r}\frac{\beta_2}{\beta_1-\beta_2}-\frac{c^2}{(r+\alpha-\sigma^2)X_1}\frac{\beta_2+1}{\beta_1-\beta_2}\right]\\
=&M_1(K)+\frac{c^{1-\beta_1}}{4\gamma(\beta_1-\beta_2)}\left(\frac{2\beta_2}{r}-\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2+1}{r+\alpha-\sigma^2}\right)\\
=&\frac{c^{1-\beta_1}[1-(1-2\gamma K)^{1+\beta_1}]}{4\gamma(\beta_1-\beta_2)}F(\beta_2).
\end{aligned}
$$

From (2.18), we get $N(K)$ as

$$
\begin{aligned}
N(K)=&M_1(K)X_2^{\beta_1-\beta_2}+M_2+X_2^{-\beta_2}\left[\frac{X_2}{4\gamma(r-\alpha)}-\frac{2c}{4\gamma r}+\frac{c^2}{4\gamma(r+\alpha-\sigma^2)X_2}\right.\\
&\left.+\frac{cK}{r}-\frac{(1-\gamma K)K}{r-\alpha}\right]\\
=&M_2+\frac{X_2^{-\beta_2}}{\beta_1-\beta_2}\left\{\frac{1}{4\gamma}\left[\frac{X_2(\beta_2-1)}{r-\alpha}+\frac{c^2(\beta_2+1)}{(r+\alpha-\sigma^2)X_2}-\frac{2c\beta_2}{r}\right]+\frac{cK\beta_2}{r}\right.\\
&\left.+\frac{(1-\beta_2)(1-\gamma K)KX_2}{r-\alpha}\right\}+X_2^{-\beta_2}\left\{\frac{1}{4\gamma}\left[\frac{X_2}{r-\alpha}-\frac{2c}{r}+\frac{c^2}{(r+\alpha-\sigma^2)X_2}\right]\right.\\
&\left.+\frac{cK}{r}-\frac{(1-\gamma K)KX_2}{r-\alpha}\right\}\\
=&M_2+\frac{X_2^{-\beta_2}}{\beta_1-\beta_2}\left\{\frac{1}{4\gamma}\left[\frac{X_2}{r-\alpha}(\beta_1-1)-\frac{2c\beta_1}{r}+\frac{c^2}{(r+\alpha-\sigma^2)X_2}(1+\beta_1)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c\beta_1 K}{r}+\frac{(1-\gamma K)KX_2}{r-\alpha}(1-\beta_1)\Bigg\} \\
=&M_2-\frac{X_2^{1-\beta_1}(1-2\gamma K)^2}{4\gamma(\beta_1-\beta_2)}\left(\frac{2\beta_1}{r}-\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1+1}{r+\alpha-\sigma^2}\right) \\
=&\frac{c^{1-\beta_2}[1-(1-2\gamma K)^{1+\beta_2}]}{4\gamma(\beta_1-\beta_2)}F(\beta_1)\,. \qquad (2.23)
\end{aligned}
$$

From the additional proof for Proposition 2.2, $\beta_1 > 0$, $\beta_2 < -1$, $F(\beta_1) < 0$ and $F(\beta_2) > 0$, implying $L(K) > 0$, $M_1(K) < 0$, $M_2(K) < 0$ and $N(K) > 0$.

Proof of Proposition 2.2 The proof of Proposition 2.2 consists of two parts. The first part derives the optimal investment timing and investment capacity for producing below and producing up to capacity right after the investment. The second part derives conditions when the firm will produce up to or below capacity right after the investment.

Derivation of optimal investment timing and capacity First, for any $X > 0$ find the optimal value of the investment capacity, $K(X)$, that maximizes the option value minus the cost of investment $V(X,K) - \delta K$. Then the optimal investment timing X^* is derived by using this optimal value. For $X < X^*$, let the value of the investment option in the continuation region be AX^{β_1}. According to value matching and smooth pasting conditions at X^*, we have

$$
\begin{cases}
AX^{*\beta_1} & = V(X^*,K(X^*)) - \delta K(X^*)\,, \\
\beta_1 AX^{*\beta_1-1} & = \frac{d}{dX}[V(X,K(X)) - \delta K(X)]\big|_{X=X^*}\,,
\end{cases}
$$

so X^* is a solution of the equation

$$
V(X,K(X)) - \delta K(X) = \frac{X}{\beta_1}\frac{d}{dX}[V(X,K(X)) - \delta K(X)] = \frac{X}{\beta_1}\frac{\partial V(X,K(X))}{\partial X}, \qquad (2.24)
$$

because

$$
\frac{\partial V(X,K(X)) - \delta K(X)}{\partial K}\frac{dK(X)}{dX} = 0.
$$

- If the firm does not produce right after the investment, then $K(X)$ should maximize $V(X,K) - \delta K$, which is

$$
\frac{c^{1-\beta_1}X^{\beta_1}[1-(1-2\gamma K)^{1+\beta_1}]}{4\gamma(\beta_1-\beta_2)}F(\beta_2) - \delta K.
$$

The first order condition implies

$$\frac{(1+\beta_1)\,F(\beta_2)\,c^{1-\beta_1}\,(1-2\gamma K(X))^{\beta_1}}{2(\beta_1-\beta_2)}X^{\beta_1}-\delta=0.$$

Then

$$K(X)=\frac{1}{2\gamma}\left[1-\frac{c}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right]. \tag{2.25}$$

The second order partial derivative of $V(X,K)$ with respect to K is negative[6], so there is a global maximum for $V(X,K)-\delta K$ when the firm does not produce right after the investment.

Determine the optimal investment timing X^* according to (2.24), then X^* is the solution for the following equation,

$$\begin{aligned}&\frac{c^{1-\beta_1}X^{\beta_1}[1-(1-2\gamma K(X))^{1+\beta_1}]}{4\gamma(\beta_1-\beta_2)}F(\beta_2)-\delta K(X)\\=&\frac{c^{1-\beta_1}X^{\beta_1}[1-(1-2\gamma K(X))^{1+\beta_1}]}{4\gamma(\beta_1-\beta_2)}F(\beta_2),\end{aligned}$$

which is equivalent to $K(X^*)=0$, contradicting to the assumption that firm invests but does not produce. So if the firm invests, the firm produces right after the investment.

- If the firm produces below capacity right after the investment, then the option value of the project is

$$V(X,K)=M_1(K)X^{\beta_1}+M_2X^{\beta_2}+\frac{X}{4\gamma(r-\alpha)}-\frac{2c}{4\gamma r}+\frac{c^2}{4\gamma(r+\alpha-\sigma^2)X}$$

with $M_1(K)$, M_2 as in (2.22) and (2.21). Letting the first order partial derivative of $V(X,K)-\delta K$ with respect to K equal 0 gives $K(X)$, the same as (2.25). Because the second order partial derivative of $V(X,K)-\delta K$ with respect to K is negative, there is a global maximum at $K(X)$.

Next, we determine the optimal investment timing X^*. If (2.24) has admissible solu-

[6] From additional proof for Proposition 2.2, $\beta_1>1$, $\beta_2<0$, $F(\beta_1)<0$, and $F(\beta_2)>0$.

tions, then we get

$$
\begin{aligned}
& M_1\left(K\left(X^*\right)\right) X^{*\beta_1} + M_2 X^{*\beta_2} + \frac{X^*}{4\gamma(r-\alpha)} - \frac{2c}{4\gamma r} \\
& + \frac{c^2}{4\gamma(r+\alpha-\sigma^2)X^*} - \delta K\left(X^*\right) \\
= & \frac{X^*}{\beta_1}\left[\frac{\partial V\left(X^*, K\left(X^*\right)\right)}{\partial X} + \frac{\partial V\left(X^*, K\left(X^*\right)\right)}{\partial K}\frac{dK\left(X^*\right)}{dX}\right] \\
= & \frac{X^*}{\beta_1}\left[\beta_1 M_1\left(K\left(X^*\right)\right) X^{*\beta_1-1} + \beta_2 M_2 X^{*\beta_2-1} + \frac{1}{4\gamma(r-\alpha)} - \frac{c^2}{4\gamma(r+\alpha-\sigma^2)X^{*2}}\right] \\
= & M_1\left(K\left(X^*\right)\right) X^{*\beta_1} + \frac{\beta_2}{\beta_1} M_2 X^{*\beta_2} + \frac{X^*}{4\gamma\beta_1\left(r-\alpha\right)} - \frac{c^2}{4\gamma\beta_1\left(r+\alpha-\sigma^2\right)X^*}.
\end{aligned}
$$

So X^* should satisfy the implicit expression

$$
\begin{aligned}
& \frac{\beta_1-\beta_2}{\beta_1} M_2 X^{*\beta_2} - \delta K\left(X^*\right) \\
& + \frac{1}{4\gamma}\left[\frac{\beta_1-1}{\beta_1}\frac{X^*}{r-\alpha} - \frac{2c}{r} + \frac{\beta_1+1}{\beta_1}\frac{c^2}{(r+\alpha-\sigma^2)X^*}\right] = 0.
\end{aligned}
$$

In case the derived $K\left(X^*\right)$ is such that $K\left(X^*\right) \leq \frac{X^*-c}{2\gamma X^*}$, i.e. the capacity is not bigger than the optimal output, then it contradicts to that the firm produces below capacity right after the investment. Thus, the firm would not invest for this case.

- If the firm produces up to capacity right after the investment, then the value of the project is

$$
V\left(X,K\right) = N\left(K\right) X^{\beta_2} + \frac{\left(1-\gamma K\right)K}{r-\alpha} X - \frac{cK}{r},
$$

where $N\left(K\right)$ is as in (2.23). The first order condition of $V\left(X,K\right) - \delta K$ with respect to K implies that the optimal value for capacity, $K\left(X\right)$, should implicitly satisfy

$$
\frac{\left(\beta_2+1\right)F\left(\beta_1\right)}{2\left(\beta_1-\beta_2\right)}\frac{\left(1-2\gamma K(X)\right)^{\beta_2}}{c^{\beta_2-1}} X^{\beta_2} + \frac{1-2\gamma K\left(X\right)}{r-\alpha} X - \frac{c}{r} - \delta = 0. \tag{2.26}
$$

In order to check the second order partial derivative of $V\left(X,K\right) - \delta K$ with respect to K, we let

$$
\mathcal{F}\left(X,K\right) = \frac{dN\left(K\right)}{dK} X^{\beta_2} + \frac{1-2\gamma K}{r-\alpha} X - \frac{c}{r} - \delta
$$

$$= \frac{F(\beta_1)\, c^{1-\beta_2}(\beta_2+1)}{2(\beta_1-\beta_2)}(1-2\gamma K)^{\beta_2} X^{\beta_2} + \frac{1-2\gamma K}{r-\alpha} X - \frac{c}{r} - \delta,$$

and from Appendix B,

$$\frac{\partial \mathcal{F}(X,K)}{\partial K} < 0.$$

So the second order partial derivative of $V(X,K) - \delta K$ with respective to K is negative, implying if equation (2.26) has an admissible solution, there is a maximum $V(X,K(X)) - \delta K(X)$. If (2.26) does not have any admissible solution, then $V(X,K) - \delta K$ is increasing or decreasing with K, and the firm would not invest for this case. We can rule out the increasing case, because it implies more capacity is better. Particularly, capacity that is bigger than $(X-c)/(2\gamma X)$ is better. This suggests the firm should invest for the case of producing below capacity right after the investment. For the decreasing case, it implies that the optimal investment capacity is 0, we can also rule out the decreasing case.

If (2.24) has admissible solutions, then the optimal investment threshold X^* is the solution of the following equation,

$$\begin{aligned} &N(K(X))X^{\beta_2} + \frac{(1-\gamma K(X))K(X)}{r-\alpha}X - \frac{cK(X)}{r} - \delta K(X) \\ &= \frac{X}{\beta_1}\left[\beta_2 N(K(X))X^{\beta_2-1} + \frac{(1-\gamma K(X))K(X)}{r-\alpha}\right]. \end{aligned}$$

Rearranging terms gives that X^* implicitly satisfies

$$\begin{aligned} &\frac{\beta_1-\beta_2}{\beta_1} N(K(X^*))X^{*\beta_2} + \frac{\beta_1-1}{\beta_1}\frac{(1-\gamma K(X^*))K(X^*)X^*}{r-\alpha} \\ &- \frac{cK(X^*)}{r} - \delta K(X^*) = 0. \end{aligned}$$

If this equation does not give any admissible solution or gives a solution that is smaller than c, or $K(X^*) > (X^*-c)/(2\gamma X^*)$, then the firm would not invest in this case.

Derivation of conditions for producing up to or below capacity right after the investment If the firm produces below capacity right after the investment (Region 2), then for $X \geq c$, it holds that

$$K(X) = \frac{1}{2\gamma} - \frac{c}{2\gamma X}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}} > \frac{X-c}{2\gamma X},$$

which is equivalent to

$$\frac{2\delta\,(\beta_1-\beta_2)}{c\,(1+\beta_1)\,F\,(\beta_2)}<1.$$

For any $X \geq c$, right after the investment, the firm either produces below capacity or up to capacity. So the firm produces up to capacity right after the investment if

$$\frac{2\delta\,(\beta_1-\beta_2)}{c\,(1+\beta_1)\,F\,(\beta_2)}\geq 1.$$

At the boundary of Region 2 and 3, the equality holds. For Region 2, we get the optimal value for investment capacity at the boundary is

$$K\,(X)=\frac{X-c}{2\gamma X}.$$

The optimal value of investment capacity at the boundary for Region 3 is the solution to (2.26) when

$$2\delta\,(\beta_1-\beta_2)=c\,(1+\beta_1)\,F\,(\beta_2)\,. \tag{2.27}$$

Then (2.26) can be written as

$$\begin{aligned}
0=&\frac{c(\beta_2+1)F(\beta_1)}{2(\beta_1-\beta_2)}\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}+\frac{X-2X\gamma K}{r-\alpha}-\frac{c}{r}-\delta\\
=&\left[\frac{c(\beta_2+1)F(\beta_1)}{2(\beta_1-\beta_2)}-\delta\right]\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}+\delta\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}\\
&+\frac{X-2X\gamma K}{r-\alpha}-\frac{c}{r}-\delta\\
=&\frac{c(\beta_2+1)F\,(\beta_1)-2(\beta_1-\beta_2)\delta}{2(\beta_1-\beta_2)}\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}\\
&+\delta\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}+\frac{X-2X\gamma K}{r-\alpha}-\frac{c}{r}-\delta\\
=&\frac{c\,(\beta_2+1)\,F\,(\beta_1)-c\,(\beta_1+1)\,F\,(\beta_2)}{2\,(\beta_1-\beta_2)}\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}\\
&+\delta\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}+\frac{X-2X\gamma K}{r-\alpha}-\frac{c}{r}-\delta\\
=&\frac{(\beta_1-\beta_2)(2\alpha\sigma^2-2\alpha^2-2\alpha r)}{2\,(\beta_1-\beta_2)}\frac{c}{r\,(r-\alpha)\,(r+\alpha-\sigma^2)}\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}\\
&+\delta\frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}}+\frac{X-2X\gamma K}{r-\alpha}-\frac{c}{r}-\delta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\alpha c}{r(r-\alpha)} \frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}} + \delta \frac{(X-2X\gamma K)^{\beta_2}}{c^{\beta_2}} + \frac{X-2X\gamma K}{r-\alpha} - \frac{c}{r} - \delta \\
&= \left(\frac{c}{r} - \frac{c}{r-\alpha} + \delta\right)\left(\frac{X-2X\gamma K}{c}\right)^{\beta_2} + \frac{X-2X\gamma K}{r-\alpha} - \frac{c}{r} - \delta \\
&= \left(\frac{c}{r} + \delta\right)\left[\left(\frac{X-2X\gamma K}{c}\right)^{\beta_2} - 1\right] + \frac{X-2X\gamma K}{r-\alpha}\left[1 - \left(\frac{X-2X\gamma K}{c}\right)^{\beta_2 - 1}\right].
\end{aligned}
$$

$K(X) = \frac{X-c}{2\gamma X}$ is a solution for this equation, implying there is smooth transfer from Region 2 to Region 3.

Next, we determine $\bar{\sigma}$ such that equation (2.27) holds as illustrated in Figure 2.2. Substituting β_1, β_2 and $F(\beta_2)$ into (2.27) gives

$$
c\left(2r\bar{\sigma}^2 + 2\alpha^2 - \alpha\bar{\sigma}^2\right) = \left[2\delta r(r-\alpha) - \alpha c\right]\sqrt{\left(\bar{\sigma}^2 - 2\alpha\right)^2 + 8r\bar{\sigma}^2}. \tag{2.28}
$$

Two cases are considered:

(a) $2\delta r(r-\alpha) \le \alpha c$ (or $\alpha \ge \frac{2\delta r^2}{c+2\delta r}$). According to (2.28), for all $\sigma > 0$ such that $r+\alpha > \sigma^2$, then

$$
\left[2\delta r(r-\alpha) - \alpha c\right]\sqrt{\left(\sigma^2 - 2\alpha\right)^2 + 8r\sigma^2} \le 0 < c\left(2r\sigma^2 + 2\alpha^2 - \alpha\sigma^2\right),
$$

which implies

$$
2\delta(\beta_1 - \beta_2) < c(1+\beta_1)F(\beta_2),
$$

and it is Region 2 defined.

(b) $2\delta r(r-\alpha) > \alpha c$ (or $\alpha < \frac{2\delta r^2}{c+2\delta r}$). Then (2.28) becomes

$$
\left(\left(\bar{\sigma}^2 - 2\alpha\right)^2 + 8r\bar{\sigma}^2\right)\left(2\delta r(r-\alpha) - \alpha c\right)^2 = c^2\left(2r\bar{\sigma}^2 + 2\alpha^2 - \alpha\bar{\sigma}^2\right)^2,
$$

which can also be written as

$$
\left(\Lambda - (2r-\alpha)^2\right)\bar{\sigma}^4 + 4\left(\Lambda - \alpha^2\right)(2r-\alpha)\bar{\sigma}^2 + 4\Lambda\alpha^2 - 4\alpha^4 = 0, \tag{2.29}
$$

with

$$
\Lambda = \left(\frac{2\delta r(r-\alpha) - \alpha c}{c}\right)^2.
$$

The discriminant for (2.29) is

$$\Delta = 64\Lambda r\left(\Lambda - \alpha^2\right)(r-\alpha),$$

and the possible solutions for $\bar{\sigma} > 0$ are supposed to satisfy either of the following

$$\bar{\sigma}_1^2 = \frac{-2\left(\Lambda-\alpha^2\right)(2r-\alpha) - 4\sqrt{r\Lambda\left(\Lambda-\alpha^2\right)(r-\alpha)}}{\Lambda-(2r-\alpha)^2};$$
$$\bar{\sigma}_2^2 = \frac{-2\left(\Lambda-\alpha^2\right)(2r-\alpha) + 4\sqrt{r\Lambda\left(\Lambda-\alpha^2\right)(r-\alpha)}}{\Lambda-(2r-\alpha)^2}.$$

Then we have the following subcases.

- If $0 < \Lambda < \alpha^2$, which is $\alpha > \frac{r^2\delta}{c+r\delta}$, then $\Delta < 0$ and (2.29) has no solution for $\bar{\sigma}^2$. Then for all $\sigma > 0$ with $r+\alpha > \sigma^2$,

$$\left(\Lambda-(2r-\alpha)^2\right)\sigma^4 + 4\left(\Lambda-\alpha^2\right)(2r-\alpha)\sigma^2 + 4\Lambda\alpha^2 - 4\alpha^4 < 0,$$

  which implies
$$2\delta(\beta_1-\beta_2) < c(1+\beta_1)F(\beta_2).$$

  So, it is Region 2 defined.

- If $\alpha^2 \leq \Lambda < (2r-\alpha)^2$, which is equivalent to $r - \frac{c}{\delta} < \alpha \leq \frac{\delta r^2}{c+\delta r}$, then $\Delta \geq 0$, and it holds that $\bar{\sigma}_1^2 \leq 0$ and $\bar{\sigma}_2^2 \geq 0$. So there is one solution for $\bar{\sigma} > 0$ and $\bar{\sigma} = \bar{\sigma}_2$. For any $\sigma > 0$ with $\sigma^2 < r+\alpha$, Region 3 is defined when $0 < \sigma \leq \bar{\sigma}$ and Region 2 is defined when $\sigma > \bar{\sigma}$.

- If $\Lambda > (2r-\alpha)^2$, which is $\alpha < r - \frac{c}{\delta}$, then $\bar{\sigma}_1^2 < 0$ and $\bar{\sigma}_2^2 < 0$. So there is no solution for $\bar{\sigma} > 0$, and for all $\sigma > 0$ with $\sigma^2 < r+\alpha$, we have

$$\left(\Lambda-(2r-\alpha)^2\right)\sigma^4 + 4\left(\Lambda-\alpha^2\right)(2r-\alpha)\sigma^2 + 4\Lambda\alpha^2 - 4\alpha^4 > 0,$$

  which implies
$$2\delta(\beta_1-\beta_2) > c(1+\beta_1)F(\beta_2).$$

  It is only Region 3 that is defined.

- If $\Lambda = (2r-\alpha)^2$, then $\delta\,(r-\alpha) = c$ and

$$\begin{aligned}
&[2\delta r\,(r-\alpha)-\alpha c]^2\left[\left(\sigma^2-2\alpha\right)^2+8r\sigma^2\right]-c^2\left(2r\sigma^2+2\alpha^2-\alpha\sigma^2\right)^2\\
=&c^2(2r-\alpha)^2\left[\left(\sigma^2-2\alpha\right)^2+8r\sigma^2\right]-c^2\left[(2r-\alpha)\sigma^2+2\alpha^2\right]^2\\
=&c^2(2r-\alpha)^2(\sigma^2-2\alpha)^2+8rc^2\sigma^2(2r-\alpha)^2-c^2\sigma^4(2r-\alpha)^2\\
&-4\alpha^2\sigma^2c^2(2r-\alpha)-4c^2\alpha^4\\
=&4c^2(2r-\alpha)^2(\alpha^2-\alpha\sigma^2+2r\sigma^2)-4c^2\alpha^2(\alpha^2-\alpha\sigma^2+2r\sigma^2)\\
=&16rc^2(r-\alpha)(\alpha^2-\alpha\sigma^2+2r\sigma^2)>0.
\end{aligned}$$

It implies that

$$2\delta\,(\beta_1-\beta_2) > c\,(1+\beta_1)\,F\,(\beta_2),$$

and Region 3 is defined.

Summarizing the above cases, it can be concluded that when $\alpha > \delta r^2/(c+\delta r)$, it is always Region 2 that is defined. When $r-c/\delta < \alpha \le \delta r^2/(c+\delta r)$, there exists $\bar{\sigma} > 0$ such that if $\bar{\sigma}^2 < r+\alpha$, then it is Region 3 for $\sigma \le \bar{\sigma}$ and Region 2 for $\sigma > \bar{\sigma}$; if $\bar{\sigma}^2 \ge r+\alpha$, then it is always Region 3. When $\alpha \le r-c/\delta$, then it is Region 3 that is defined.

Additional proof of $\partial\mathcal{F}(X,K)/\partial K < 0$ for Proposition 2.2 Before we check the sign of $\partial\mathcal{F}(X,K)/\partial K$, we first look at the signs for β_1, β_2 and $F(\beta_1)$. $r > 0$ and the assumption $r > \alpha$ imply $(\frac{1}{2}-\frac{\alpha}{\sigma^2})^2+\frac{2r}{\sigma^2} > (\frac{1}{2}+\frac{\alpha}{\sigma^2})^2$, thus $\beta_1 > 1$ and $\beta_2 < -1$ if $\frac{1}{2}+\frac{\alpha}{\sigma^2} > 0$. If $\frac{1}{2}+\frac{\alpha}{\sigma^2} \le 0$, then $\frac{\alpha}{\sigma^2} \le -\frac{1}{2}$. The assumption $r+\alpha-\sigma^2 > 0$ implies $\frac{2r}{\sigma^2} > 2-\frac{2\alpha}{\sigma^2}$. Then

$$\begin{aligned}
\beta_2 =&\frac{1}{2}-\frac{\alpha}{\sigma^2}-\sqrt{(\frac{1}{2}-\frac{\alpha}{\sigma^2})^2+\frac{2r}{\sigma^2}}\\
<&\frac{1}{2}-\frac{\alpha}{\sigma^2}-\sqrt{(\frac{1}{2}-\frac{\alpha}{\sigma^2})^2+2-\frac{2\alpha}{\sigma^2}}\\
=&\frac{1}{2}-\frac{\alpha}{\sigma^2}-\sqrt{\frac{9}{4}-\frac{3\alpha}{\sigma^2}+(\frac{\alpha}{\sigma^2})^2}\\
=&\frac{1}{2}-\frac{\alpha}{\sigma^2}+\frac{\alpha}{\sigma^2}-\frac{3}{2}\\
=&-1.
\end{aligned}$$

Thus, it can be concluded that $\beta_2 < -1$.

Recall that

$$F(\beta)=\frac{\beta\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)+r\left(2\alpha-\sigma^2\right)}{r\left(r-\alpha\right)\left(r+\alpha-\sigma^2\right)},$$

with $r(r-\alpha)(r+\alpha-\sigma^2)>0$ since $r>\alpha$ and $r+\alpha>\sigma^2$.

If $2\alpha<\sigma^2$, then

$$\begin{aligned}r(r-\alpha)(r+\alpha-\sigma^2)F'(\beta)&=2\alpha\sigma^2-r\sigma^2-2\alpha^2\\&<2\alpha\sigma^2+\alpha\sigma^2-\sigma^4-2\alpha^2\\&=-\sigma^4+3\alpha\sigma^2-2\alpha^2\\&=-\left(\sigma^2-\alpha\right)\left(\sigma^2-2\alpha\right)\\&<0;\end{aligned}$$

and if $\sigma^2<2\alpha$, then

$$\begin{aligned}r(r-\alpha)(r+\alpha-\sigma^2)F'(\beta)&=2\alpha\sigma^2-r\sigma^2-2\alpha^2\\&=\alpha\sigma^2-r\sigma^2+\alpha\sigma^2-2\alpha^2\\&=\sigma^2\left(\alpha-r\right)+\alpha\left(\sigma^2-2\alpha\right)\\&<0.\end{aligned}$$

So $F'(\beta)<0$. Define β_0 such that $F\left(\beta_0\right)=0$, then

$$\beta_0=\frac{r\left(2\alpha-\sigma^2\right)}{r\sigma^2+2\alpha^2-2\alpha\sigma^2}.$$

Because $F\left(\beta\right)$ decreases with β, if we can compare the values for β_0, β_1 and β_2, then it would be easy to get the signs for $F\left(\beta_1\right)$ and $F\left(\beta_2\right)$. Let

$$G\left(\beta\right)=\frac{\sigma^2}{2}\beta^2+\left(\alpha-\frac{\sigma^2}{2}\right)\beta-r.$$

β_1 and β_2 are the intersection points of $G\left(\beta\right)$ and the β-axis. If we can show that $G\left(\beta_0\right)<0$, then $\beta_1>\beta_0>\beta_2$, $F\left(\beta_1\right)<0$ and $F(\beta_2)>0$.

$$G\left(\beta_0\right)=\frac{\sigma^2}{2}\beta_0^2+\left(\alpha-\frac{\sigma^2}{2}\right)\beta_0-r$$

$$
\begin{aligned}
&= \frac{1}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\left\{\frac{r^2\sigma^2\left(2\alpha-\sigma^2\right)^2}{2}-r\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2\right.\\
&\quad\left.+\frac{r}{2}\left(2\alpha-\sigma^2\right)^2\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]\right\}\\
&= \frac{r}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\left\{\left(2\alpha-\sigma^2\right)^2\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)\right.\\
&\quad\left.-\left(r\sigma^2-\alpha\sigma^2+\alpha^2+\alpha^2-\alpha\sigma^2\right)^2\right\}\\
&= \frac{r}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\left\{\left(2\alpha-\sigma^2\right)^2\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)\right.\\
&\quad\left.-\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)^2-2\left(\alpha^2-\alpha\sigma^2\right)\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)-\left(\alpha^2-\alpha\sigma^2\right)^2\right\}\\
&= \frac{r\left[\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)\left(-r\sigma^2-\alpha\sigma^2+\alpha^2+\sigma^4\right)-\left(\alpha^2-\alpha\sigma^2\right)^2\right]}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\\
&= \frac{r\left[\left(\alpha^2-\alpha\sigma^2\right)^2-r^2\sigma^4+\sigma^4\left(r\sigma^2-\alpha\sigma^2+\alpha^2\right)-\left(\alpha^2-\alpha\sigma^2\right)^2\right]}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\\
&= \frac{r\sigma^4(r-\alpha)}{\left[(r-\alpha)\sigma^2+2\alpha^2-\alpha\sigma^2\right]^2}\left[\sigma^2-(r+\alpha)\right].
\end{aligned}
$$

Because $r>\alpha$ and $\sigma^2<r+\alpha$, we get $G(\beta_0)<0$. Thus, $F(\beta_1)<0$, and $F(\beta_2)>0$. Next, we check the sign for $\partial\mathcal{F}(X,K)/\partial K$.

$$
\frac{\partial\mathcal{F}(X,K)}{\partial K}=-\frac{\beta_2\gamma F(\beta_1)X^{\beta_2}(\beta_2+1)}{(\beta_1-\beta_2)c^{\beta_2-1}}(1-2\gamma K)^{\beta_2-1}-\frac{2\gamma X}{r-\alpha},
$$

where

$$
-\frac{\beta_2\gamma F(\beta_1)(\beta_2+1)}{(\beta_1-\beta_2)c^{\beta_2-1}}>0.
$$

Because Region 3 is defined such that $X\geq\frac{c}{1-2\gamma K}$, we have

$$
\begin{aligned}
\frac{\partial\mathcal{F}(X,K)}{\partial K} &\leq \gamma X\left[-\frac{\beta_2 F(\beta_1)(\beta_2+1)}{\beta_1-\beta_2}-\frac{2}{r-\alpha}\right]\\
&= \frac{\gamma X}{r-\alpha}\left[-\frac{\beta_2(\beta_2+1)}{(\beta_1-\beta_2)}\frac{\beta_1\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)+r\left(2\alpha-\sigma^2\right)}{r\left(r+\alpha-\sigma^2\right)}-2\right]\\
&= \frac{\gamma X}{r-\alpha}\left[\frac{\beta_2+1}{\beta_1-\beta_2}\frac{\frac{2r}{\sigma^2}\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)-r\beta_2\left(2\alpha-\sigma^2\right)}{r\left(r+\alpha-\sigma^2\right)}-2\right]\\
&= \frac{\gamma X}{(\beta_1-\beta_2)r(r-\alpha)\left(r+\alpha-\sigma^2\right)}\left[(\beta_2+1)\frac{2}{\sigma^2}\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)\right.
\end{aligned}
$$

$$-\beta_2(\beta_2+1)\left(2\alpha-\sigma^2\right)-2(\beta_1-\beta_2)\left(r+\alpha-\sigma^2\right)\Bigg]$$

and

$$
\begin{aligned}
&(\beta_2+1)\frac{2}{\sigma^2}\left(2\alpha\sigma^2-r\sigma^2-2\alpha^2\right)-\beta_2(\beta_2+1)\left(2\alpha-\sigma^2\right)-2(\beta_1-\beta_2)\left(r+\alpha-\sigma^2\right)\\
=&(2\beta_2+2)\left(2\alpha-r-\frac{2\alpha^2}{\sigma^2}\right)-\beta_2^2\left(2\alpha-\sigma^2\right)-\beta_2\left(2\alpha-\sigma^2\right)\\
&\quad-2\beta_1\left(r+\alpha-\sigma^2\right)+2\beta_2\left(r+\alpha-\sigma^2\right)\\
=&2\beta_2\left(2\alpha-\frac{2\alpha^2}{\sigma^2}-\frac{\sigma^2}{2}\right)-\beta_2^2\left(2\alpha-\sigma^2\right)+2\left(2\alpha-r-\frac{2\alpha^2}{\sigma^2}\right)-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&2\beta_2\left[\frac{\alpha}{\sigma^2}\left(-2\alpha+\sigma^2\right)+\frac{1}{2}\left(2\alpha-\sigma^2\right)\right]-\beta_2^2\left(2\alpha-\sigma^2\right)+2(\alpha-r)+\frac{2\alpha}{\sigma^2}\left(\sigma^2-2\alpha\right)\\
&-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&2\beta_2\left(2\alpha-\sigma^2\right)\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)-\beta_2^2\left(2\alpha-\sigma^2\right)+2(\alpha-r)+\frac{2\alpha}{\sigma^2}\left(\sigma^2-2\alpha\right)\\
&-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&-\frac{\beta_2}{\sigma^2}\left(2\alpha-\sigma^2\right)^2-\beta_2^2\left(2\alpha-\sigma^2\right)+2(\alpha-r)-\frac{2\alpha}{\sigma^2}\left(2\alpha-\sigma^2\right)-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&\frac{\left(2\alpha-\sigma^2\right)^2}{\sigma^2}\left(\frac{2\alpha-\sigma^2}{2\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\right)+2(\alpha-r)-\frac{2\alpha}{\sigma^2}\left(2\alpha-\sigma^2\right)\\
&-\left(2\alpha-\sigma^2\right)\left[2\left(\frac{\sigma^2-2\alpha}{2\sigma^2}\right)^2+\frac{2r}{\sigma^2}+\frac{2\alpha-\sigma^2}{\sigma^2}\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\right]\\
&-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&\frac{\left(2\alpha-\sigma^2\right)^3}{2\sigma^4}+\frac{\left(2\alpha-\sigma^2\right)^2}{\sigma^2}\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}-\frac{2\alpha}{\sigma^2}\left(2\alpha-\sigma^2\right)-\frac{\left(2\alpha-\sigma^2\right)^3}{2\sigma^4}\\
&-\frac{2r\left(2\alpha-\sigma^2\right)}{\sigma^2}-\frac{\left(2\alpha-\sigma^2\right)^2}{\sigma^2}\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}-2\beta_1\left(r+\alpha-\sigma^2\right)+2(\alpha-r)\\
=&2(\alpha-r)-\frac{2\left(2\alpha-\sigma^2\right)}{\sigma^2}(r+\alpha)-2\beta_1\left(r+\alpha-\sigma^2\right)\\
=&2(\alpha-r)-\frac{2\left(2\alpha-\sigma^2\right)}{\sigma^2}(r+\alpha)-2\left(r+\alpha-\sigma^2\right)\left(\frac{\sigma^2-2\alpha}{2\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\right)\\
=&2(\alpha-r)-\frac{2\alpha-\sigma^2}{\sigma^2}\left(r+\alpha+\sigma^2\right)-2\left(r+\alpha-\sigma^2\right)\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}.
\end{aligned}
$$

Since $\beta_1 > 0$, $\beta_2 < 0$, $r - \alpha > 0$ and $r + \alpha - \sigma^2 > 0$, we conclude that if $2\alpha - \sigma^2 \geq 0$, then $\partial \mathcal{F}(X,K)/\partial K < 0$. However, if $\sigma^2 - 2\alpha > 0$, then we continue with the expression above and get

$$\begin{aligned}
& 2(\alpha - r) - \frac{2\alpha - \sigma^2}{\sigma^2}\left(r + \alpha + \sigma^2\right) - 2\left(r + \alpha - \sigma^2\right)\sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \\
& < 2(\alpha - r) - \frac{2\alpha - \sigma^2}{\sigma^2}\left(r + \alpha + \sigma^2\right) - 2\left(r + \alpha - \sigma^2\right)\sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2} \\
& = 2(\alpha - r) - \frac{2\alpha - \sigma^2}{\sigma^2}\left(r + \alpha + \sigma^2\right) - 2\left(r + \alpha - \sigma^2\right)\frac{\sigma^2 - 2\alpha}{2\sigma^2} \\
& = 2(\alpha - r) + 2\left(\sigma^2 - 2\alpha\right) = -2\left(r + \alpha - \sigma^2\right) < 0.
\end{aligned}$$

Thus, we conclude that

$$\frac{\partial \mathcal{F}(X,K)}{\partial K} < 0.$$

CHAPTER 3

Strategic Capacity Investment under Uncertainty with Volume Flexibility

This chapter considers investment decisions in an uncertain and competitive framework, with a first investor, the leader, always producing up to full capacity and a second investor, the follower, capable of adjusting output levels within the constraint of installed capacity. Both firms need to decide on the investment timing and the investment capacity levels. The main findings are as follows. Compared to a situation where the follower always produces up to full capacity, the leader has a larger incentive to accommodate a flexible follower. This is because the leader also benefits from the follower's volume flexibility. Due to the first mover advantage, the leader's value is higher than the follower's value, despite the follower's technological advantage in flexibility. This chapter is based on Wen (2017).

3.1 Introduction

Uncertainty is a main characteristic of the business environment nowadays. The technology advancement has shortened product life cycles, increased product variety, and indulged more demanding consumers. This contributes to the uncertainty in consumer demand and poses challenges on the manufacturing firms. The ability to produce to the least cost is no longer enough. The capability to absorb demand fluctuations has become an important competitive issue. Flexibility is considered as an adaptive response to the environmental uncertainty (Gupta and Goyal, 1989). Browne et al. (1984) define eight different types of flexibilities, among which the volume flexibility is described as “the ability to operate an FMS (Flexible Manufacturing Systems) profitably at different production volumes." Sethi and Sethi (1990) further describe volume flexibility of a manufacturing system as “its ability to be operated profitably at different overall output levels." According to Beach et al. (2000), utilizing flexibility presents performance-related benefits. Numerous studies have argued for the importance of volume flexibility, see Jack and Raturi (2002). For instance, Goyal and Netessine (2011) show that volume flexibility may help the firm combat the product demand uncertainty. In a monopolistic market, Hagspiel et al. (2016) and Wen et al. (2017) analyze the volume flexibility's influences on a monopolistic investor's investment decision and show that it increases the value of the investment. In a competitive setting, an important question for the investors would be how the flexibility influences investment decisions and the investors' strategic interactions.

This chapter considers volume flexibility in a homogenous good market with exogenous firm roles. Demand is linear and subject to stochastic shocks, which follow a geometric Brownian motion process. There are two firms that decide on entering the duopoly market by investing in a production plant. More specifically, they have to decide about the timing and the investment capacity. One firm, the leader, has dedicated technology. The other firm, i.e., the follower who invests second, has volume flexibility. The leader always has to produce up to capacity and has a first mover advantage. The follower can adjust the output levels according to market demand. One can easily find both dedicated and flexible firms in the electricity market: a nuclear power station is dedicated and a fossil fuel power station is flexible. According to Goyal and Netessine (2007), a firm may find it difficult to produce below capacity due to fixed costs associated with, for example, labor, commitment to suppliers and production ramp-up[1]. A surprising outcome of our research is that, since the market price is affected by the follower's flexible output, the leader benefits from the follower's flexibility when market demand is low. This is because the follower reduces the output quantity in such a case.

Our analysis starts with a market where no firms are active. Then two domains on market sizes are identified for the leader, with one domain where it is optimal to deter the entry of the flexible follower and the other one where it is optimal to accommodate the entry. We show that entry deterrence domain increases with uncertainty. This result is the same as in Huisman and Kort (2015), where the follower is dedicated. Besides, we find that compared to a dedicated follower, the leader is less likely to deter a flexible follower. This is because when there is uncertainty about market demand, both the leader and the flexible follower tend to wait for more information about the market and invest later. For the entry deterrence strategy, the leader has an incentive to overinvest to deter the entry of the follower[2]. Incapable of adjusting to the instant market demand, the leader is more vulnerable to the negative demand shocks. For the follower, the volume flexibility yields higher values and thus motivates to invest earlier compared with a dedicated follower. This results in a shorter monopoly period for the leader and diminishes the attractiveness of entry deterrence compared to the case where the follower is dedicated. Furthermore, compared to a dedicated follower, it is more likely for the leader to accommodate a flexible follower.

[1] I do not model these issues explicitly in this chapter.

[2] Overinvesting refers to that a firm invests more capacity as the first investor than when investing simultaneously with the other firm at a predetermined point of time.

For the accommodation strategy, the two firms invest at the same time, so the incentive to overinvest in order to deter the follower's entry disappears. The market price reacts to the follower's output adjustment, and this diminishes the leader's vulnerability to demand uncertainty. The incentive to overinvest in order to reduce the capacity size of the flexible follower and to benefit from the follower's output adjustment is still strong. This makes accommodation of the flexible follower more attractive to the leader.

We also find that in a fast growing market, the flexible follower produces below capacity right after investment. While in a slowly growing or shrinking market, the flexible follower produces up to capacity right after investment. In the intermediate case, the flexible follower produces up to capacity right after investment when uncertainty is low and below capacity when uncertainty is high. These findings are the same as that for the flexible monopolist by Wen et al. (2017). The strategic interactions between the leader and the flexible follower do not influence these results. Moreover, there is free riding on the follower's flexibility since the volume flexibility affects market prices, and thus enlarges the profitability of the leader. So, the flexible follower cannot fully capture the innovative benefits from the technology advancement. However, this does not diminish the follower's incentive to invest in the volume flexibility technology, because it still generates a larger value for the follower whether the leader chooses and entry deterrence or entry accommodation strategy.

The duopoly model with volume flexibility first contributes to the research stream of monopolistic volume flexibility investment combining investment timing and capacity determination, by Dangl (1999), Hagspiel et al. (2016), and Wen et al. (2017). The general result is that flexibility leads to an increase in the optimal installed capacity and project value. The influence of flexibility on investment timing depends on two effects, with one effect that higher value motivates a flexible firm to invest earlier and the other effect that larger installed capacity motivates it to invest later. This chapter shows that flexibility affects the flexible follower in a similar way as it affects the flexible monopolist. Its influence on the leader depends on the leader's competition strategy, and the dedicated leader also gets a higher value when playing the accommodation strategy.

In this chapter, firms not only make decisions about capacities, but also about investment timings in the continuous time setting. It contributes to the literature of capacity choices with volume flexibility in a competitive framework using discrete time models. Gabszewicz and Poddar (1997) study a two-stage model with capacity choice in the first stage and capacity constrained quantity competition in the second stage and show that the firms choose the certainty-equivalent Cournot capacity. If the second stage is a capacity-constrained price competition instead of quantity competition, Reynolds and Wilson (2000) find that symmetric equilibrium does not exist in pure strategies for capacity choices if demand is sufficient volatile. Besanko and Doraszelski (2004) consider two types of competition: quantity competition and price competition in each period of an infinite time horizon. Quantity competition results in an industry structure of equal-sized firms, while price competition results in unequal-sized firms.

Besides the economics literature, volume flexibility is also studied in operations management. For example, Anupindi and Jiang (2008) consider the volume flexibility in a three-stage framework: capacity choice in the first stage, production decisions in the second stage and pricing decisions in the third stage. Flexible firms can make production decisions when demand is observed. Under competition, they find that firms choose to be inflexible for multiplicative demand shocks, while flexible for additive demand shocks. In a two-product setting with demand uncertainty for both products, Goyal and Netessine (2011) introduce volume flexibility and find that volume flexibility combats aggregate demand uncertainty for the two products. Current research on volume flexibility focuses more on the capacity choices and adopts discrete time models in the analysis. For every dynamic period, the firm needs to decide whether and how much to invest conditional on the available information at the beginning of the period, see for instance Besanko and Doraszelski (2004) and Besanko et al. (2010). In these two papers, the purpose is to analyze the firm sizes in market equilibria. By using a continuous time model, this chapter analyzes the decision on both investment timing and investment capacity. More specifically, this research analyzes the influence of volume flexibility on the timing of market entry. In a competitive setting, the first investor has a larger incentive to accommodate than to deter the entry of the second investor, given the second investor has volume flexibility. This is due to the fact that volume flexibility combats demand uncertainty for both investors in the market, similarly as that proposed by Goyal and Netessine (2011) for two products.

The duopoly model with flexibility in this research also extends the literature on entry deterrence and entry accommodation investment. According to Lieberman and Montgomery (1988), the first investor's investment serves as a commitment to maintain a high level of production output, which is a price cut threat to decrease entrant's profit. Spence (1977) and Dixit (1980) study preemptive commitment by constructing static investment models and show that entry can be deterred by installing excess capacity to make a new entrant unprofitable. Maskin (1999) introduces uncertainty and obtains the same conclusion. By discrete time models, Reynolds (1987) shows that the equilibrium capacity choice is a decreasing function of the current rival capacity. Besanko et al. (2010) argue that preemption is more likely when the products in the market have low heterogeneity and there is uncertainty about the entrant's exact cost/benefit of capacity addition/withdrawal. In this chapter, the products are homogeneous, time is continuous, and there is uncertainty about the market demand. The asymmetric firm roles of a dedicated leader and a flexible follower, are direct extensions of symmetric firm roles of both a dedicated leader and a dedicated follower by Huisman and Kort (2015). This chapter shows that in a continuous time setting, excess capacity can help the first investor to deter the entry of the second investor and the second investor's optimal capacity decreases with the first investor's capacity. However, the first investor's optimal investment timing and capacity are independent of the second investor's volume flexibility. Moreover, when the second investor has volume flexibility, the entry accommodation is more likely when the products are homogenous and there is market demand uncertainty. For the given incumbent's decisions, Yang and Zhou (2007) show that it is impossible for the incumbent with excess capacity to deter the potential entrant who holds the option to entry forever. This result is supported also by Huisman and Kort (2015), who not only consider the deterrence of the potential entrant, but also the possibility of accommodation of the potential entrant. They construct the domains on market sizes of entry deterrence and accommodation strategy for the duopoly setting where the only difference between investors is the cost advantage for the first investor. In this chapter, the difference between the two investors is that the leader always produces up to full capacity, while the follower can adjust the output within the capacity constraint. Similar to Huisman and Kort (2015), this chapter also constructs the domains for the leader's entry strategies. By comparing situation of volume flexibility with situation of no volume flexibility, this chapter shows that the first investor has less incentive to deter the entry of the second investor if the second investor has volume flexibility.

This chapter is organized as follows. Section 3.2 describes the duopoly investment problem. Section 3.3 analyses the flexible follower's optimal investment decision. The dedicated leader's optimal investment decision is derived in Section 3.4. In Section 3.5, the influence of flexibility on the leader and the follower is analysed. Section 3.6 concludes.

3.2 Model Setup

Consider a framework where two firms can invest in production capacity to enter a market or serve a particular demand. Of the two firms, the follower (second investor) can access volume flexibility technology and adjust output levels up to the installed capacity after the investment. The leader (first investor) has no access to such technology and can only produce at full capacity level. Denote by $K_D \geq 0$ and $K_F \geq 0$ the capacity of the flexible follower and dedicated leader, respectively. For both firms, the unit cost for capacity investment is $\delta > 0$ and the unit cost for production is $c > 0$. The price at time $t \geq 0$ is $p(t)$, given by the inverse demand function

$$p(t) = X(t)\left[1 - \gamma(q_D(t) + q_F(t))\right],$$

where $\gamma > 0$ is a constant, $q_D(t)$ and $q_F(t)$ denote the production output for the dedicated and flexible firm at time t, respectively, and the uncertainty in demand, $\{X(t)|t \geq 0\}$, follows a geometric Brownian Motion (GBM) process

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW_t,$$

in which $X(0) > 0$, α is the trend parameter, $\sigma > 0$ is the volatility parameter, and dW_t is the increment of a Wiener process. The inverse linear demand function has among others been adopted by Pindyck (1988) and Huisman and Kort (2015). Both firms are risk neutral and have a discount rate of r, which is assumed to be larger than α, the trend of GBM $X(t)$, and larger than $\sigma^2 - \alpha$, the trend for GBM $\{1/X(t)\}$. This is to prevent that it is optimal for the firms to always delay the investment (see Dixit and Pindyck, 1994). From now on we drop the argument of time whenever there can be no misunderstanding.

3.3 Flexible Follower's Optimal Investment Decision

The leader is assumed to be already in the market when the flexible follower makes investment decisions. Given level $X(t) = X$ and the leader's investment capacity K_D, denote $\pi_F(X, K_D, K_F)$ as the profit for the flexible follower after investing in capacity K_F. The follower is flexible and can adjust its production between 0 and the invested capacity K_F. The output by the flexible follower maximises the profit flow, which is

$$\pi_F(X, K_D, K_F) = \max_{0 \le q_F \le K_F} \{X[1 - \gamma(K_D + q_F)] - c\} q_F.$$

Given $0 \le K_D < 1/\gamma$, the optimal production level for the follower is

$$q_F(X, K_D, K_F) = \begin{cases} 0 & 0 < X < \frac{c}{1-\gamma K_D}, \\ \frac{X-c}{2\gamma X} - \frac{K_D}{2} & X \ge \frac{c}{1-\gamma K_D} \text{ and } K_F > \frac{X-c}{2\gamma X} - \frac{K_D}{2}, \\ K_F & X \ge \frac{c}{1-\gamma K_D} \text{ and } K_F \le \frac{X-c}{2\gamma X} - \frac{K_D}{2}. \end{cases} \tag{3.1}$$

The corresponding profit flow is

$$\pi_F(X, K_D, K_F) = \begin{cases} 0 & 0 < X < \frac{c}{1-\gamma K_D}, \\ \frac{(X-c-\gamma X K_D)^2}{4\gamma X} & X \ge \frac{c}{1-\gamma K_D} \text{ and } K_F > \frac{X-c}{2\gamma X} - \frac{K_D}{2}, \\ (X - c - \gamma X K_D) K_F - K_F^2 \gamma X & X \ge \frac{c}{1-\gamma K_D} \text{ and } K_F \le \frac{X-c}{2\gamma X} - \frac{K_D}{2}. \end{cases} \tag{3.2}$$

The flexible follower's investment decision is solved as an optimal stopping problem and can be formalised as

$$\sup_{T \ge 0, K_F \ge 0} E\left[\int_T^\infty \pi_F(X(t), K_D, K_F) \exp(-rt)dt - \delta K_F \exp(-rT) \Big| X(0)\right],$$

conditional on the available information at time 0, where T is the time when the flexible follower invests, and K_F is the acquired capacity at time T. Denote by $V_F(X,K_D,K_F)$ the value for the flexible follower, it satisfies the Bellman equation

$$rV_F = \pi_F + \frac{1}{dt}E[dV_F]. \tag{3.3}$$

Similar to the previous chapter, applying Ito's Lemma, substituting and rewriting lead to the following differential equation (see also, e.g., Dixit and Pindyck (1994))

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_F(X,K_D,K_F)}{\partial X^2} + \alpha X \frac{\partial V_F(X,K_D,K_F)}{\partial X} - rV_F(X,K_D,K_F) + \pi_F(X,K_D,K_F) = 0. \tag{3.4}$$

Substituting (3.2) into (3.4) and employing value matching and smooth pasting for $X = c/(1-\gamma K_D)$ and $X = c/(1-\gamma K_D - 2\gamma K_F)$ yield the follower's value after investment as

$$V_F(X,K_D,K_F) = \begin{cases} L(K_D,K_F)X^{\beta_1} & 0 < X < \frac{c}{1-\gamma K_D}, \\ M_1(K_D,K_F)X^{\beta_1} + M_2(K_D)X^{\beta_2} & \\ +\frac{(1-\gamma K_D)^2 X}{4\gamma(r-\alpha)} - \frac{c(1-\gamma K_D)}{2\gamma r} + \frac{c^2}{4\gamma X\left(r+\alpha-\sigma^2\right)} & X \geq \frac{c}{1-\gamma K_D} \text{ and } K_F > \frac{X-c}{2\gamma X} - \frac{K_D}{2}, \\ N(K_D,K_F)X^{\beta_2} - \frac{cK_F}{r} + \frac{K_F - \gamma K_D K_F - \gamma K_F^2}{r-\alpha}X & X \geq \frac{c}{1-\gamma K_D} \text{ and } K_F \leq \frac{X-c}{2\gamma X} - \frac{K_D}{2}, \end{cases} \tag{3.5}$$

in which

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{3.6}$$

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < -1. \tag{3.7}$$

The expressions and derivation of $L(K_D,K_F)$, $M_1(K_D,K_F)$, $M_2(K_D)$, $N(K_D,K_F)$, as well as all the proofs of the following propositions, lemmas and corollaries can be found in the appendix. If $K_D = 0$, the model reduces to monopoly case. The follower does not produce right after the investment for $0 < X < c/(1-\gamma K_D)$. Thus, $L(K_D,K_F)X^{\beta_1}$ is positive and represents the option value to start producing in the future as soon as X reaches $c/(1-\gamma K_D)$. $M_1(K_D,K_F)X^{\beta_1}$ is negative and corrects for the fact that if X reaches $c/(1-\gamma K_D - 2\gamma K_F)$, the follower's output will be constrained by the installed capacity level. $M_2(K_D)X^{\beta_2}$ is negative and corrects for the positive quadratic form of cash flows such that when X drops below $c/(1-\gamma K_D)$, the follower would temporarily suspend the production. $N(K_D,K_F)X^{\beta_2}$ is positive and describes the option value that if demand decreases, e.g., X drops below

$c/(1-\gamma K_D - 2\gamma K_D)$, the follower produces below full capacity. The optimal investment decision is found in two steps. First, given K_D and the level of X, the optimal value of K_F is found by maximising $V_F(X, K_D, K_F) - \delta K_F$, which yields $K_F(X, K_D)$. Second, the optimal investment threshold $X_F^*(K_D)$ for the follower can be derived. The two steps are summarised in the following proposition, where $\bar{\sigma}$ is such that

$$\bar{\sigma}^2 = \frac{-2\left(\Lambda - \alpha^2\right)(2r-\alpha) + 4\sqrt{r\Lambda\left(\Lambda-\alpha^2\right)(r-\alpha)}}{\Lambda - (2r-\alpha)^2}, \tag{3.8}$$

with $\Lambda = \left(\frac{2\delta r(r-\alpha)-\alpha c}{c}\right)^2$. $\bar{\sigma} > 0$ is a value of the drift parameter that determines if the follower produces below or up to capacity right after investment. $\bar{\sigma}$ is only defined for $r - c/\delta < \alpha \leq \delta r^2/(c+\delta r)$.

Proposition 3.1. *Given the dedicated firm has already invested in capacity $K_D \in [0, 1/\gamma)$, there are two possibilities regarding the follower's investment decisions:*

1. *Suppose $\alpha > \delta r^2/(c+\delta r)$, or both $r - c/\delta < \alpha \leq \delta r^2/(c+\delta r)$ and $\sigma > \bar{\sigma}$. Then the follower does not produce up to capacity right after investment. For any $X \geq c/(1-\gamma K_D)$, the optimal capacity $K_F(X, K_D)$ that maximizes $V(X, K_D, K_F) - \delta K_F$ is*

$$K_F(X, K_D) = \frac{1}{2\gamma}\left\{1 - \gamma K_D - \frac{c}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}, \tag{3.9}$$

and the optimal investment threshold $X_F^*(K_D)$ satisfies

$$\frac{c^{1-\beta_2}(1-\gamma K_D)^{1+\beta_2}F(\beta_1)X^{\beta_2}}{4\gamma\beta_1} + \frac{1}{4\gamma}\left[\frac{\beta_1-1}{\beta_1}\frac{(1-\gamma K_D)^2 X}{r-\alpha} - \frac{2c(1-\gamma K_D)}{r}\right.$$
$$\left. + \frac{\beta_1+1}{\beta_1}\frac{c^2}{X(r+\alpha-\sigma^2)}\right] - \delta K_F(X, K_D) = 0. \tag{3.10}$$

If $X(0) < X_F^*(K_D)$, then the optimal capacity of the follower is $K_F^*(K_D) = K_F(X_F^*(K_D), K_D)$. If $X(0) \geq X_F^*(K_D)$, then the follower invests immediately at $t = 0$ with capacity $K_F^*(K_D) = K_F(X(0), K_D)$.

2. *Suppose $\alpha \leq r - c/\delta$, or both $r - c/\delta < \alpha \leq \delta r^2/(c+\delta r)$ and $\sigma \leq \bar{\sigma}$. Then the follower produces up to capacity right after the investment. For any $X \geq c/(1-\gamma K_D)$, the optimal capacity $K_F(X, K_D)$ that maximizes $V(X, K_D, K_F) - \delta K_F$ satisfies*

$$\frac{F(\beta_1)c^{1-\beta_2}(1+\beta_2)(1-2\gamma K_F - \gamma K_D)^{\beta_2}}{2(\beta_1-\beta_2)}X^{\beta_2} + \frac{1-2\gamma K_F - \gamma K_D}{r-\alpha}X - \frac{c}{r} - \delta = 0, \tag{3.11}$$

and the optimal investment threshold $X_F^*(K_D)$ satisfies

$$\frac{c^{1-\beta_2}F(\beta_1)X^{\beta_2}}{4\gamma\beta_1}\left[(1-\gamma K_D)^{1+\beta_2}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_2}\right]$$
$$+\frac{(\beta_1-1)X}{\beta_1}\frac{K_F-\gamma K_D K_F-\gamma K_F^2}{r-\alpha}-\frac{cK_F}{r}-\delta K_F=0, \tag{3.12}$$

with $K_F = K_F(X, K_D)$. If $X(0) < X_F^*(K_D)$, then the optimal capacity of the follower is $K_F^*(K_D) = K_F(X_F^*(K_D), K_D)$. If $X(0) \geq X_F^*(K_D)$, then the follower invests immediately at $t = 0$ with capacity $K_F^*(K_D) = K_F(X(0), K_D)$.

From Proposition 3.1, the influence of the leader's investment capacity on the follower's investment decision is concluded in Corollary 3.1.

Corollary 3.1. *The dedicated leader's capacity level K_D influences the follower's investment decision: If the leader invests more, the follower invests later and invests less.*

This is intuitive because the leader always produces up to capacity after investment, and the more the leader invests, the smaller market share is left for the flexible follower regardless of whether the follower produces below or up to capacity after investment. When deciding on the investment capacity, the follower takes the future market demand into consideration. Thus, the decline in the market share decreases the follower's investment capacity. For a given GBM level X, the leader's increased capacity also lowers the market price. This delays the follower's investment because the follower needs to wait for a higher market price.

3.4 Dedicated Leader's Optimal Investment Decision

When deciding on the investment timing and capacity, the leader takes the follower's investment decision into account. For a given level of $X(t) = X$ at time t, suppose the leader invests at t with capacity size K_D. Because the follower's optimal threshold $X_F^*(K_D)$ increases with K_D, the leader can invest in a larger (smaller) capacity size such that the follower invests later (earlier). Assume there exists a critical capacity size for the leader, $\hat{K}_D(X)$, such that the follower's optimal threshold satisfies $X_F^*(\hat{K}_D) = X$. This critical capacity size $\hat{K}_D(X)$ can be derived from (3.10) as to satisfy

$$\frac{c^{1-\beta_2}X^{\beta_2}(1-\gamma K_D)^{1+\beta_2}F(\beta_1)}{2\beta_1}+\frac{\beta_1-1}{2\beta_1}\frac{X(1-\gamma K_D)^2}{r-\alpha}-\frac{c(1-\gamma K_D)}{r}$$
$$+\frac{\beta_1+1}{2\beta_1}\frac{c^2}{X(r+\alpha-\sigma^2)}-\delta(1-\gamma K_D)+\frac{c\delta}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}=0. \tag{3.13}$$

From Corollary 3.1, it can be concluded that if $K_D \leq \hat{K}_D(X)$, then $X \geq X_F^*(K_D)$, implying that the follower invests at the same time with the leader. If $K_D > \hat{K}_D(X)$, then $X < X_F^*(K_D)$, which implies that the follower invests later than the leader. The former case corresponds to the leader's entry accommodation strategy and the latter corresponds to its entry deterrence strategy, as described by Huisman and Kort (2015). In the following analysis, the dedicated leader's value function is derived and the leader's optimal entry accommodation and entry deterrence strategy are characterized as local maximum to the leader's value maximization problem, which is formulated at the moment of investment as

$$\sup_{K_D \geq 0} E\left[\int_0^T (K_D(1-\gamma K_D)X(t) - cK_D)\, e^{-rt}\mathrm{d}t \right.$$
$$\left. + \int_T^\infty (K_D(1-\gamma K_D - \gamma q_F(X,K_D,K_F))X(t) - cK_D)\, e^{-rt}\mathrm{d}t - \delta K_D \,\middle|\, X(0) = X\right],$$

where T is the moment that the flexible follower invests. It holds that $T > 0$ under the entry deterrence strategy and $T = 0$ under the entry accommodation strategy.

To derive the leader's value function, first take a look at the leader's profit after the follower invests. Because of volume flexibility, the follower might not produce, produce below, and produces up to capacity after investment. For each of these three cases, given GBM level X and the invested capacity size K_D, the leader's profit flow $\pi_D(X,K_D)$, when both firms are active in the market, is

$$\pi_D(X,K_D) = \begin{cases} K_D(1-\gamma K_D)X - cK_D & \text{if } 0 < X < \frac{c}{1-\gamma K_D}, \\ \frac{K_D}{2}(X - c - \gamma X K_D) & \text{if } X \geq \frac{c}{1-\gamma K_D} \text{ and } K_F^*(K_D) > \frac{X-c}{2\gamma X} - \frac{K_D}{2}, \\ XK_D[1-\gamma(K_D + K_F^*(K_D))] - cK_D & \text{if } X \geq \frac{c}{1-\gamma K_D} \text{ and } K_F^*(K_D) \leq \frac{X-c}{2\gamma X} - \frac{K_D}{2}. \end{cases}$$

Applying Ito's Lemma, substituting and rewriting leads to the following differential equation (see, e.g., Dixit and Pindyck (1994))

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_D(X,K_D)}{\partial X^2} + \alpha X \frac{\partial V_D(X,K_D)}{\partial X} - rV_D(X,K_D) + \pi_D(X,K_D) = 0.$$

Substituting π_D into this differential equation and employing value matching and smooth pasting at $X = c/(1-\gamma K_D)$ and $X = c/(1-\gamma K_D - 2\gamma K_F^*(K_D))$ give the value of the leader after the follower's investment as

$$V_D(X,K_D)=\begin{cases}\mathcal{L}(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r} & \text{if } 0\le X<\frac{c}{1-\gamma K_D},\\ \mathcal{M}_1(K_D)X^{\beta_1}+\mathcal{M}_2(K_D)X^{\beta_2} & \\ +\frac{XK_D(1-\gamma K_D)}{2(r-\alpha)}-\frac{cK_D}{2r} & \text{if } X\ge\frac{c}{1-\gamma K_D} \text{ and } K_F^*(K_D)>\frac{X-c}{2\gamma X}-\frac{K_D}{2},\\ \mathcal{N}(K_D)X^{\beta_2}-\frac{cK_D}{r} & \\ +\frac{K_D(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}X & \text{if } X\ge\frac{c}{1-\gamma K_D} \text{ and } K_F^*(K_D)\le\frac{X-c}{2\gamma X}-\frac{K_D}{2}.\end{cases} \tag{3.14}$$

The derivation and expressions of $\mathcal{L}(K_D)$, $\mathcal{M}_1(K_D)$, $\mathcal{M}_2(K_D)$ and $\mathcal{N}(K_D)$ with their signs can be found in the appendix.

For $0\le X<c/(1-\gamma K_D)$, the demand is very small and the follower's production is temporarily suspended. However, incapable of adjusting to the market demand, the leader still produces at full capacity. In the leader's value function, $\mathcal{L}(K_D)X^{\beta_1}$ measures the decrease in the leader's value when the follower resumes production in the future. This happens as soon as X becomes larger than $c/(1-\gamma K_D)$. For $c/(1-\gamma K_D)\le X<c/(1-\gamma K_D-2\gamma K_F^*(K_D))$, e.g., $X\ge c/(1-\gamma K_D)$ and $K_F^*(K_D)>(X-c)/(2\gamma X)-K_D/2$, the follower produces below capacity right after investment. $\mathcal{M}_1(K_D)X^{\beta_1}$ corrects for the fact that if X reaches $c/(1-\gamma K_D-2\gamma K_F^*(K_D))$, then the production of the follower is constrained by the follower's installed capacity, hence the value of the leader increases. The term $\mathcal{M}_2(K_D)X^{\beta_2}$ denotes the decrease in the leader's option value. This is due to the fact that when X falls below $c/(1-\gamma K_D)$, the market demand becomes so small that the follower suspends production, whereas the leader still produces at full capacity, which results in negative profit. For $X\ge c/(1-\gamma K_D-2\gamma K_F^*(K_D))$, e.g., $X\ge c/(1-\gamma K_D)$ and $K_F^*(K_D)\le(X-c)/(2\gamma X)-K_D/2$, the follower produces up to capacity right after investment. The term $\mathcal{N}(K_D)X^{\beta_2}$ corrects for the fact that if X drops below $c/(1-\gamma K_D-2\gamma K_F^*(K_D))$, then the follower produces below capacity, and the value of the leader would increase.

In what follows, we analyze the leader's strategies for two cases, i.e., the follower produces below and up to capacity right after investment. This is because from Wen et al. (2017), the flexible firm always produces right after investment. Before the follower invests, the leader's value function consists of two parts with one part from the monopolistic profit flow, and the other part correcting the changes in values when the follower invests and the leader loses its monopoly privilege. When the leader invests at X, let the dedicated leader's value before the follower invests take the following form

$$V_D(X,K_D)=\mathcal{B}(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r},$$

where $\mathcal{B}(K_D)$ has different expressions and will be derived for the two cases[3]. The leader's value after the follower invests is shown in (3.14). Then in every case two leader's strategies are analyzed, namely the entry deterrence strategy and the entry accommodation strategy. We begin with the case of below capacity, then we analyze the case of up to capacity.

- The flexible follower produces below capacity right after the investment when $\alpha > \delta r^2/(c+\delta r)$, or both $r - c/\delta < \alpha \le \delta r^2/(c+\delta r)$ and $\sigma > \bar{\sigma}$.

The leader's value function given that the leader invests at X before and after the follower's entry is as follows

$$V_D(X,K_D) = \begin{cases} \mathcal{B}_1(K_D)X^{\beta_1} + \frac{K_D(1-\gamma K_D)}{r-\alpha}X - \frac{cK_D}{r} & X < X_F^*(K_D), \\ \mathcal{M}_1(K_D)X^{\beta_1} + \mathcal{M}_2(K_D)X^{\beta_2} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X - \frac{cK_D}{2r} & X \ge X_F^*(K_D), \end{cases} \tag{3.15}$$

with

$$\begin{aligned} \mathcal{B}_1(K_D) =& \mathcal{M}_1(K_D) + \mathcal{M}_2(K_D)X_F^{*\beta_2-\beta_1}(K_D) \\ & - \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X_F^{*1-\beta_1}(K_D) + \frac{cK_D}{2r}X_F^{*-\beta_1}(K_D), \end{aligned} \tag{3.16}$$

according to value matching at the follower's investment threshold $X_F^*(K_D)$, which is defined by (3.10). Intuitively, $\mathcal{B}_1(K_D)$ is negative (see appendix). It corrects for the fact that when $X(t)$ reaches $X_F^*(K_D)$, the follower enters the market, putting an end to the leader's monopolistic privilege. The leader's entry deterrence and accommodation strategies when the follower produces below capacity right after investment are described in the following proposition.

Proposition 3.2. *Suppose $\alpha > \delta r^2/(c+\delta r)$, or both $r - c/\delta < \alpha \le \delta r^2/(c+\delta r)$ and $\sigma > \bar{\sigma}$.*

(a) Entry Deterrence Strategy
The entry deterrence strategy will be considered whenever $X \in (X_1^{det}, X_2^{det})$, where X_1^{det} satisfies

$$\left[-\frac{\delta}{(1+\beta_1)F(\beta_2)}\left(\frac{\beta_2-1}{r-\alpha} - \frac{\beta_2}{r}\right) + \frac{c^{1-\beta_2}X_F^{*\beta_2}(0)}{2(\beta_1-\beta_2))}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right) \right.$$

[3] $\mathcal{B}(K_D)$ and $\mathcal{L}(K_D)$ are different. The fundamental component (see Dixit and Pindyck (1994)) in the leader's value function, $\frac{K_D(1-\gamma K_D)}{r}X - \frac{K_D}{r}$, is generated by the profit flows. $\mathcal{L}(K_D)X^{\beta_1}$ describes the deviation of $V_D(X,K_D)$ from the fundamental component due to the possibility that X will move across the boundary $\frac{c}{1-\gamma K_D}$. $\mathcal{B}(K_D)X^{\beta_1}$ describes the deviation of $V_D(X,K_D)$ from the fundamental component due to the possibility that X will move across the follower's optimal investment threshold X_F^*.

$$-\frac{X_F^*(0)}{2(r-\alpha)}+\frac{c}{2r}\Bigg]\left(\frac{X^{det}}{X_F^*(0)}\right)^{\beta_1}+\frac{X^{det}}{r-\alpha}-\frac{c}{r}-\delta=0, \tag{3.17}$$

and X_2^{det} together with $K_D^{det}(X_2^{det})$ satisfy equations (3.13) *and*

$$\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_1(K_D)(X^{det})^{\beta_1}+\frac{1-2\gamma K_D}{r-\alpha}X^{det}-\frac{c}{r}-\delta=0. \tag{3.18}$$

The optimal investment threshold X_D^{det} and investment capacity K_D^{det} under the entry deterrence strategy are

$$\begin{aligned} X_D^{det} &= \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right), \\ K_D^{det} &\equiv K_D^{det}(X_D^{det})=\frac{1}{(\beta_1+1)\gamma}, \end{aligned}$$

when $X<X_D^{det}$ and $X_D^{det}\in[X_1^{det},X_2^{det}]$. If $X_D^{det}\le X\le X_2^{det}$, in order to implement the entry deterrence strategy, the leader invests immediately at X with capacity $K_D^{det}(X)$, which satisfies (3.18)*. The value of the entry deterrence strategy is*

$$V_D^{det}(X)=\mathcal{B}_1(K_D^{det}(X))X^{\beta_1}+\frac{K_D^{det}(X)\left(1-\gamma K_D^{det}(X)\right)}{r-\alpha}X-\frac{cK_D^{det}(X)}{r}. \tag{3.19}$$

(b) Entry Accommodation Strategy
The entry accommodation strategy will be considered if $X\ge X_1^{acc}$, where X_1^{acc} and the corresponding $K_D^{acc}(X_1^{acc})$ satisfy (3.13) *and*

$$\begin{aligned}\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)(X^{acc})^{\beta_1}+\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)(X^{acc})^{\beta_2}& \\ +\frac{1-2\gamma K_D}{2(r-\alpha)}X^{acc}-\frac{c}{2r}-\delta=0.&\end{aligned} \tag{3.20}$$

The optimal investment threshold X_D^{acc} satisfies

$$\frac{c}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{\beta_1X^{acc}}{c(\beta_1+1)}\right)^{\beta_2}+\frac{(\beta_1-1)X^{acc}}{2(r-\alpha)(\beta_1+1)}-\frac{c}{2r}-\delta=0, \tag{3.21}$$

when $X<X_D^{acc}$ and $X_D^{acc}\ge X_1^{acc}$. The optimal investment capacity for the entry accommodation strategy is

$$K_D^{acc}\equiv K_D^{acc}(X_D^{acc})=\frac{1}{(\beta_1+1)\gamma}.$$

If $X\ge X_D^{acc}$, in order to implement the entry accommodation strategy, the leader invests immediately at X with capacity $K_D^{acc}(X)$ that satisfies (3.20)*. The value of the entry* *accommodation strategy is*

$$\begin{aligned} V_D^{acc}(X) =& \mathcal{M}_1(K_D^{acc}(X))X^{\beta_1} + \mathcal{M}_2(K_D^{acc}(X))X^{\beta_2} \\ &+ \frac{K_D^{acc}(X)\,(1-\gamma K_D^{acc}(X))}{2(r-\alpha)}X - \frac{cK_D^{acc}(X)}{2r}. \end{aligned} \tag{3.22}$$

A numerical example is provided to demonstrate how the leader plays against a flexible follower when the follower produces below capacity right after investment. Figure 3.1 illustrates the capacity levels $\hat{K}_D$, K_D^{det}, and K_D^{acc} as functions of X. For the given parameter values, the leader would consider the entry deterrence strategy for $X \in [X_1^{det}, X_2^{det}]$, and the entry accommodation strategy for $X \geq X_1^{acc}$. When both strategies are applicable, the dedicated leader would choose the strategy that generates higher value.

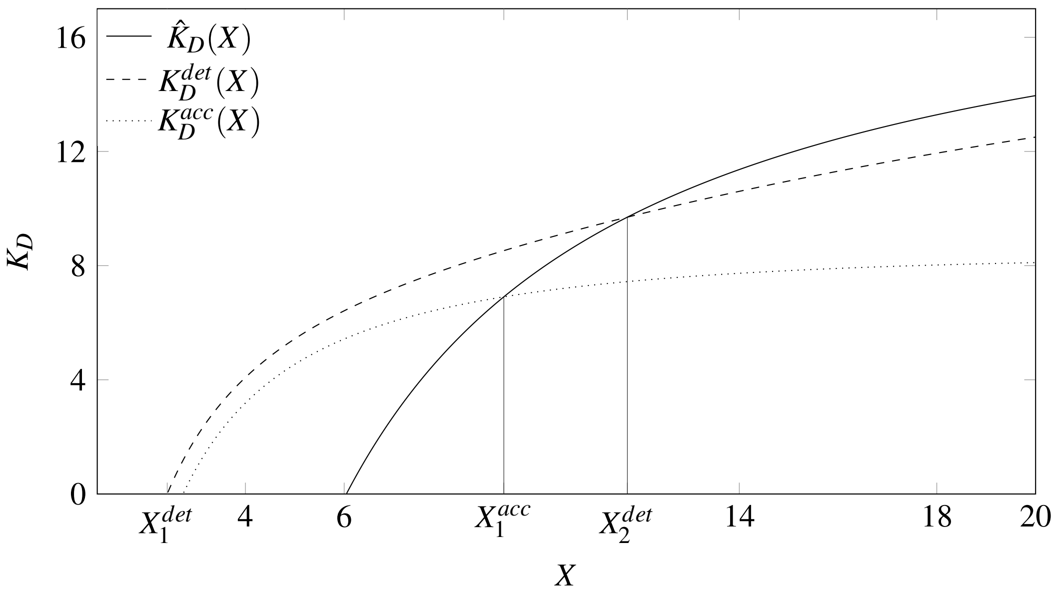


Figure 3.1: Illustration of $\hat{K}_D(X)$, $K_D^{det}(X)$, and $K_D^{acc}(X)$ when the flexible follower produces below capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

More specifically, for the given parameter values in Figure 3.1, $X_1^{det} = 2.42$, $X_2^{det} = 11.73$. The optimal threshold for entry deterrence strategy is $X_D^{det} = 6.30$. Suppose the current level of geometric Brownian motion is X. If $X < 6.30$, to delay the entry of the flexible follower, the leader waits until X reaches 6.30. For any X between 6.30 and 11.73, the leader needs to invest immediately to delay the flexible follower. For $X > 11.73$, the entry deterrence strategy is not possible because the market demand is large enough for both firms to be active. Moreover, $X_D^{acc} = 8.50$ and $X_1^{acc} = 9.23$, which implies the accommodation strategy is to invest immediately when $X(t)$ reaches 9.23. $X_D^{acc} < X_1^{acc}$ makes X_D^{acc} have no meaning for this numerical example as the demand parameter has to reach the value of X_1^{acc} to make the follower invest at the same time as the leader.

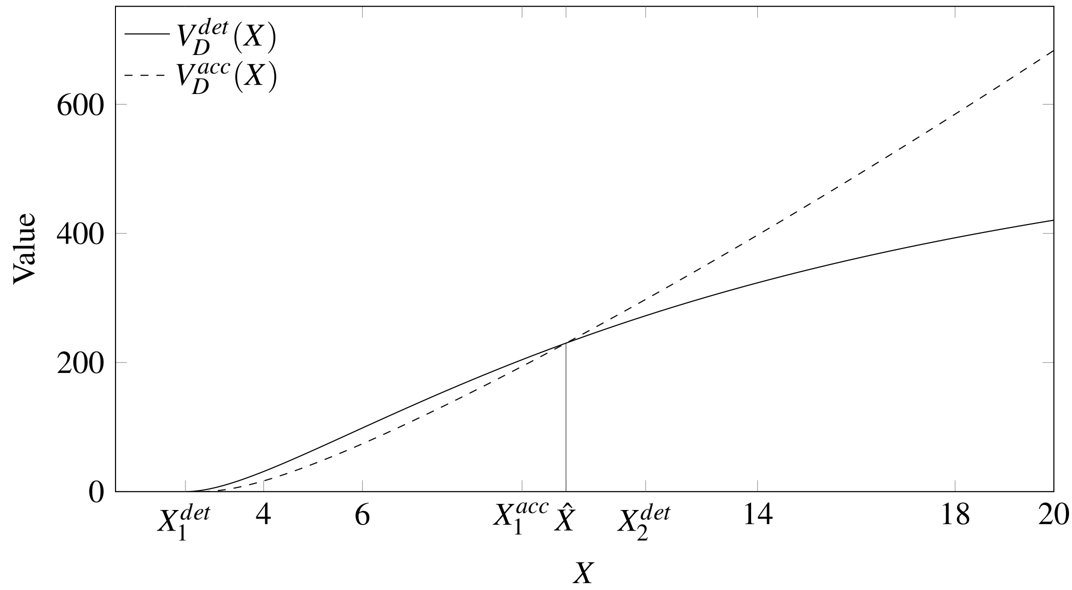


Figure 3.2: Illustration of $V_D^{det}(X)$ and $V_D^{acc}(X)$ when the flexible follower produces below capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

Figure 3.2 shows the value of the entry deterrence strategy V_D^{det} and accommodation strategy V_D^{acc} as functions of X, when the flexible follower produces below capacity right after investment. For $X_1^{det} < X < \hat{X}$, the entry deterrence strategy is chosen by the leader and the leader would invest at $X_D^{det} = 6.30$ with capacity $K_D(X^{det}) = 6.67$. For $X \geq \hat{X}$, because $\hat{X} > X_1^{acc}$, the leader would choose entry accommodation strategy by investing immediately with capacity level $K_D^{acc}(X)$.

- The flexible follower produces up to capacity right after the investment when $\alpha \leq r - c/\delta$, or both $r - c/\delta < \alpha \leq \delta r^2/(c + \delta r)$ and $\sigma \leq \bar{\sigma}$.

Similar to the case where the flexible follower produces below capacity right after investment, given that the leader invests at X, its value function before and after the follower's entry can be written as

$$V_D(X, K_D) = \begin{cases} \mathcal{B}_2(K_D)X^{\beta_1} + \frac{K_D(1-\gamma K_D)}{r-\alpha}X - \frac{cK_D}{r} & X < X_F^*(K_D), \\ \mathcal{N}(K_D)X^{\beta_2} + \frac{K_D(1-\gamma K_D - \gamma K_F^*(K_D))}{r-\alpha}X - \frac{cK_D}{r} & X \geq X_F^*(K_D), \end{cases} \tag{3.23}$$

with $\mathcal{B}_2(K_D)$

$$\mathcal{B}_2(K_D) = \mathcal{N}(K_D)X_F^{*\beta_2-\beta_1}(K_D) - \frac{\gamma K_D K_F^*(K_D)}{r-\alpha}X_F^{*1-\beta_1}(K_D), \tag{3.24}$$

according to the value matching condition at the flexible follower's investment threshold

$X_F^*(K_D)$, which is defined by (3.12).

Similar as $\mathcal{B}_1(K_D)$, $\mathcal{B}_2(K_D)$ corrects for the fact that when the follower enters the market, i.e. X reaches $X_F^*(K_D)$, it would put an end to the leader's monopoly privilege. Thus, $\mathcal{B}_2(K_D)$ is negative. Because $X_F^*(K_D)$ increases with K_D according to Corollary 3.1, it is possible for the dedicated leader to delay the entry of flexible follower through the entry deterrence strategy by investing $K_D^{det}(X) > \hat{K}_D(X)$. Otherwise, the two firms invest at the same time, implying the leader applies the entry accommodation strategy by investing $K_D^{acc} \leq \hat{K}_D(X)$. This critical size for the leader's capacity, $\hat{K}_D(X)$, can be derived from (3.12) with the follower's optimal investment capacity $K_F^*(X) \equiv K_F^*(K_D(X))$ satisfying (3.11).

The leader's investment decision under entry deterrence and accommodation strategies, when the follower produces up to capacity right after investment, are summarised in the following proposition.

Proposition 3.3. *Suppose $\alpha \leq r - c/\delta$, or both $r - c/\delta < \alpha \leq \delta r^2/(c+\delta r)$ and $\sigma \leq \bar{\sigma}$.*

(a) Entry Deterrence Strategy

The entry deterrence strategy is possible if $X \in \left(X_1^{det}, X_2^{\text{det}}\right)$. X_2^{det}, $K_D^{det}(X_2^{det})$ and $K_F^*(X_2^{det})$ satisfy (3.11), (3.12), and

$$\frac{1-\gamma K_D - \beta_1 \gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_2(K_D)(X^{det})^{\beta_1} + \frac{1-2\gamma K_D}{r-\alpha}X^{det} - \frac{c}{r} - \delta = 0. \tag{3.25}$$

X_1^{det} satisfies

$$\begin{aligned}\frac{c}{2(\beta_1-\beta_2)}\left(\frac{X^{det}}{X_F^*(0)}\right)^{\beta_1}\Bigg\{&\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[\left(\frac{X_F^*(0)}{c}\right)^{\beta_2} - \left(\frac{X_F^*(0)\,(1-2\gamma K_F^*(0))}{c}\right)^{\beta_2}\right] \\ &-\frac{\beta_1-\beta_2}{r-\alpha}\frac{2\gamma X_F^*(0)K_F^*(0)}{c}\Bigg\} + \frac{X^{det}}{r-\alpha} - \frac{c}{r} - \delta = 0,\end{aligned} \tag{3.26}$$

where $K_F^*(0)$ and $X_F^*(0)$ are such that

$$\frac{c\,(1+\beta_2)\,F(\beta_1)}{2(\beta_1-\beta_2)}\left(\frac{X_F^*(0)\,(1-2\gamma K_F^*(0))}{c}\right)^{\beta_2} + \frac{X_F^*(0)\,(1-2\gamma K_F^*(0))}{r-\alpha} - \frac{c}{r} - \delta = 0, \tag{3.27}$$

and

$$\begin{aligned}&\frac{cF(\beta_1)}{4\gamma\beta_1}\left(\frac{X_F^*(0)}{c}\right)^{\beta_2}\left(1-(1-2\gamma K_F^*(0))^{1+\beta_2}\right) + \frac{\beta_1-1}{\beta_1}\frac{X_F^*(0)\left(K_F^*(0)-\gamma K_F^{*2}(0)\right)}{r-\alpha} \\ &-\frac{cK_F^*(0)}{r} - \delta K_F^*(0) = 0.\end{aligned} \tag{3.28}$$

The optimal investment threshold X_D^{det} and the corresponding optimal capacity K_D^{det}

for the entry deterrence strategy are equal to

$$\begin{aligned} X_D^{det} &= \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right), \\ K_D^{det} &\equiv K_D^{det}(X_D^{det}) = \frac{1}{(\beta_1+1)\gamma}, \end{aligned}$$

if $X < X_D^{det}$ *and* $X_D^{det} \in [X_1^{det}, X_2^{det}]$. *If* $X_D^{det} \le X < X_2^{det}$, *in order to implement the entry deterrence strategy, the leader invests immediately at* X *with capacity* $K_D^{det}(X)$ *that satisfies* (3.25). *The value of the entry deterrence strategy is*

$$V_D^{det}(X) = \mathcal{B}_2(K_D^{det}(X))X^{\beta_1} + \frac{K_D^{det}(X)\left(1-\gamma K_D^{det}(X)\right)}{r-\alpha}X - \frac{cK_D^{det}(X)}{r}. \tag{3.29}$$

(b) *Entry Accommodation Strategy*
The entry accommodation strategy is possible if $X > X_1^{acc}$. X_1^{acc}, $K_D^{acc}(X_1^{acc})$, *and* $K_F^*(X_1^{acc})$ *satisfy* (3.11), (3.12), *and*

$$\frac{(1-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)}\mathcal{N}(K_D)(X^{acc})^{\beta_2} + \frac{X^{acc}(1-\gamma K_D-\gamma K_F^*(K_D))(1-2\gamma K_D)}{(r-\alpha)(1-\gamma K_D)} - \frac{c}{r} - \delta = 0. \tag{3.30}$$

The optimal investment threshold X_D^{acc} *satisfies*

$$\begin{aligned} &\frac{c(X^{acc})^{\beta_2}}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{c}{1-\gamma K_D^{acc}}\right)^{-\beta_2} - \left(\frac{c}{1-\gamma K_D^{acc}-2\gamma K_F^*(K_D^{acc})}\right)^{-\beta_2}\right) \\ &+ \frac{(\beta_1-1)X^{acc}}{\beta_1(r-\alpha)}(1-\gamma K_D^{acc}-\gamma K_F^*(K_D^{acc})) - \frac{c}{r} - \delta = 0, \end{aligned} \tag{3.31}$$

if $X < X_D^{acc}$ *and* $X_D^{acc} \ge X_1^{acc}$. *The optimal investment capacity for the entry accommodation strategy is*

$$K_D^{acc} \equiv K_D^{acc}(X_D^{acc}) = \frac{1}{(\beta_1+1)\gamma}.$$

If $X \ge X_D^{acc}$, *in order to implement the entry accommodation strategy, the leader invests immediately at* X *and the corresponding capacity* $K_D^{acc}(X)$ *satisfies* (3.30). *The value of the entry accommodation strategy is*

$$V_D^{acc}(X) = \mathcal{N}(K_D^{acc}(X))X^{\beta_2} + \frac{K_D^{acc}(X)\left(1-\gamma K_D-\gamma K_F^*(K_D^{acc}(X))\right)}{r-\alpha}X - \frac{cK_D^{acc}(X)}{r}. \tag{3.32}$$

Similar to the previous case, a numerical example is provided to illustrate how the leader decides on investment when the follower produces up to capacity right after investment.

Figure 3.3 gives a numerical illustration for $\hat{K}_D$, K_D^{det}, and K_D^{acc} as functions of X when the flexible follower produces up to capacity right after investment. For the given parameter

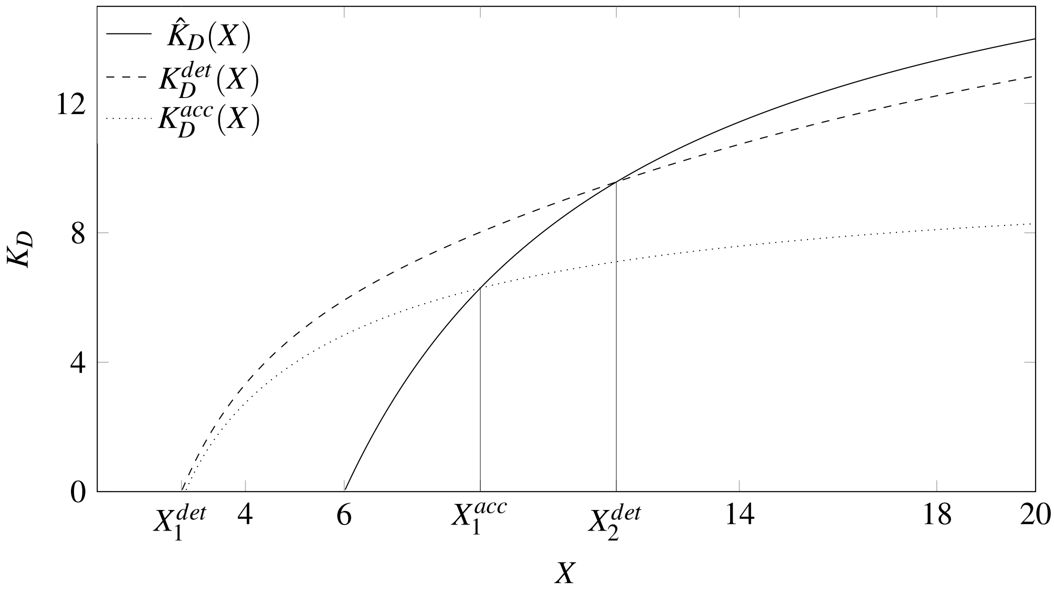


Figure 3.3: Illustration of $\hat{K}_D(X)$, $K_D^{det}(X)$, and $K_D^{acc}(X)$ when the flexible follower produces up to capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.02$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

values, the entry deterrence strategy will be considered if $X \in \left(X_1^{det}, X_2^{det}\right)$ with $X_1^{det} = 2.70$ and $X_2^{det} = 11.51$. The corresponding capacities are $K_D^{det}(X_1^{det}) = 0$ and $K_D^{det}(X_2^{det}) = 9.57$. The optimal investment threshold is $X_D^{det} = 6.28$ and the optimal capacity size is $K_D^{det} = 6.18$ if $X < 6.28$. For the entry accommodation strategy, $X_D^{acc} = 8.47$ and $K_D^{acc} = 6.18$ if $X < 8.47$. However, for this numerical example, $X_1^{acc} = 8.76$ and it holds that $X_D^{acc} < X_1^{acc}$, implying X_D^{acc} has no meaning as the demand parameter has to reach the value of X_1^{acc} to make the follower invest at the same time as the leader. Thus, in order to play the entry accommodation strategy, the leader waits until X reaches 8.76 and invests immediately.

Figure 3.4 demonstrates the values of the entry deterrence strategy $V_D^{det}(X)$ and the entry accommodation strategy V_D^{acc}(X) as functions of X when the flexible follower produces up to capacity right after investment. Again, the dedicated leader will consider the entry deterrence strategy for $X_1^{det} < X < \hat{X}$ and the entry accommodation strategy for $X \geq \hat{X}$. So to implement the entry deterrence strategy, the leader invests at $X_D^{det} = 6.28$ with capacity $K_D^{det} = 6.18$ if $X < X_D^{det}$, and invests immediately at X with capacity $K_D^{det}(X)$ if $X_D^{det} \leq X < \hat{X}$. To implement the entry accommodation strategy, the leader invests immediately at X when $X \geq \hat{X}$ with capacity $K_D^{acc}(X)$.

It can be concluded from Proposition 3.2 and 3.3 that the entry accommodation strategy is not possible for $X < X_1^{acc}$, and the entry deterrence strategy is not possible for $X > X_2^{det}$. When $X_1^{acc} < X < X_2^{det}$, the strategy that gives higher value will be chosen by the leader. Huisman and Kort (2015) has shown analytically that $X_1^{acc} < X_2^{det}$ when there is no volume flexibility for the follower. With Figure 3.5 we check numerically whether this still holds for a flexible follower. Figure 3.5 shows that departing from the default parameter values

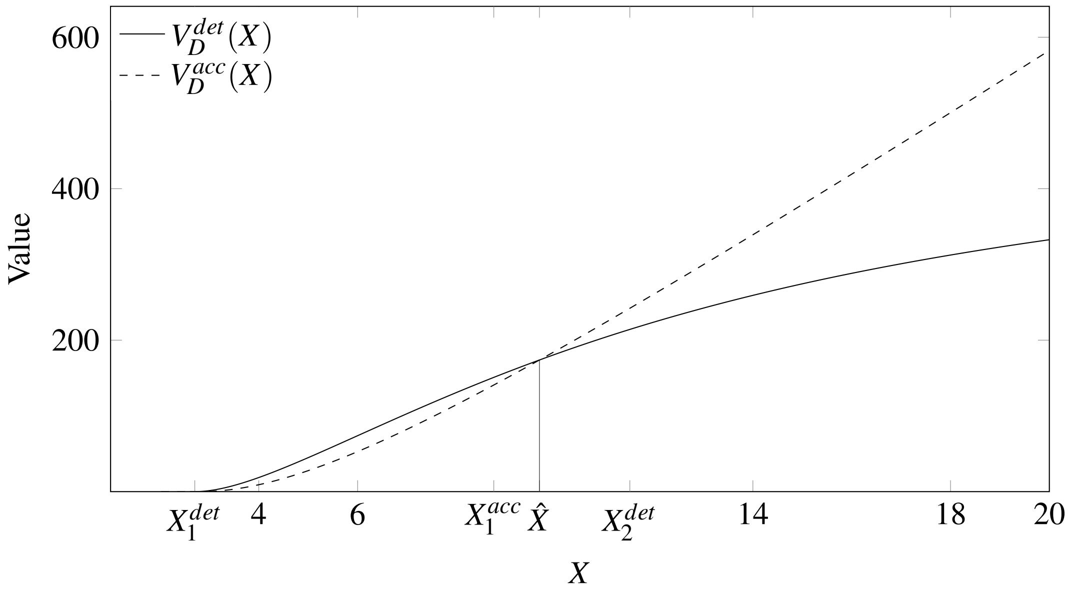


Figure 3.4: Illustration of $V_D^{det}(X)$ and $V_D^{acc}(X)$ when the flexible follower produces up to capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.02$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

$\alpha = 0.03$, $\sigma = 0.2$, $r = 0.1$, $c = 2$, $\delta = 10$, and $\gamma = 0.05$, when changing σ, α, r, c, δ and γ, X_2^{det} is always larger than X_1^{acc}. Thus, we can assume that $X_2^{det} > X_1^{acc}$ also holds for the setting with follower's volume flexibility. However, different from Huisman and Kort (2015), where $X_D^{acc} < X_1^{acc}$ always holds, the numerical analysis in Figure 3.5 shows that for significantly small α or δ, it actually holds that $X_D^{acc} > X_1^{acc}$. Note that X_D^{acc} implies that the market demand should be large enough to accommodate both firms' entry at the same time. When α is small or negative, implying that the follower produces up to full capacity right after investment, a larger market demand is required to accommodate two firms and it leads to $X_D^{acc} > X_1^{acc}$. When δ is small, i.e., it is not expensive to invest, both firms are encouraged to install larger capacities and a larger X_D^{acc} is required to hold both firms in the market. The above results are summarized in the following proposition.

Proposition 3.4. *Denote $\hat{X}$ as*

$$\hat{X} = \min\{X | X_1^{acc} < X < X_2^{det} \textit{ and } V_D^{acc}(X) = V_D^{det}(X)\}.$$

Let $X(t) = X$, the optimal investment capacity for the leader is

$$K_D^*(X) = \begin{cases} K_D^{det}(X_D^{det}) & \textit{if } 0 \leq X < X_D^{det}, \\ K_D^{det}(X) & \textit{if } X_D^{det} \leq X < \hat{X}, \\ K_D^{acc}(X_D^{acc}) \textit{ or } K_D^{det}(\hat{X}) & \textit{if } \hat{X} \leq X < X_D^{acc}, \\ K_D^{acc}(X) & \textit{if } X \geq \max\{\hat{X}, X_D^{acc}\}. \end{cases} \tag{3.33}$$

The optimal investment threshold for the leader is

$$X_D^* = \begin{cases} X_D^{det} & \text{if } 0 \leq X < X_D^{det}, \\ X & \text{if } X_D^{det} \leq X < \hat{X}, \\ X_D^{acc} \text{ or } \hat{X} & \text{if } \hat{X} \leq X < X_D^{acc}, \\ X & \text{if } X \geq \max\{\hat{X}, X_D^{acc}\}. \end{cases} \tag{3.34}$$

The optimal capacity level for the leader $K_D^*(X)$ and for the flexible follower $K_F^*(X)$ are demonstrated in Figure 3.6. For the given parameter values, $X_D^{det} = 6.3$, $X_1^{acc} = 8.3671 < X_D^{acc} = 8.3710 < \hat{X} = 9.2809$, and the flexible follower produces up to capacity right after investment. According to Proposition 3.4, if $X < \hat{X}$, the leader chooses the entry deterrence strategy. Note that for $X < X_D^{det}$, the leader is waiting to invest. Once level X_D^{det} is reached, the leader invests with capacity $K_D^{det}(X_D^{det}) = 5.7143$. Then the follower would wait until $X_F^*(K_D^{det}) = 8.3682$ is reached and invest with $K_F^*(K_D^{det}) = 4.1746$. If $X_D^{det} < X \leq \hat{X}$, the leader invests immediately at level X with $K_D^{det}(X)$, and the follower invests later at $X_F^*(K_D^{det}(X))$ with $K_F^*(K_D^{det}(X))$. If $\hat{X} < X < X_D^{acc}$, the investment happens when X reaches X_D^{acc}, and the leader applies the entry accommodation strategy. There is also investment when X decreases and reaches $\hat{X}$, where the leader implements the entry deterrence strategy. If $X > \max\{\hat{X}, X_D^{acc}\}$, the leader invests immediately at X with capacity level $K_D^{acc}(X)$. The follower invests at the same time with capacity $K_F^*(K_D^{acc}(X))$. Figure 3.6 shows that for the entry deterrence strategy, the leader's optimal investment capacity increases with X when $X < \hat{X}$. This is because as the demand becomes larger, in order to postpone the follower's entry and to prolong the monopoly privilege, the leader needs to install more capacity. According to Huisman and Kort (2015), when $X = \hat{X}$, this overinvestment can be seen as the difference in $K_D^*(X)$ at $\hat{X}$, because the entry accommodation strategy capacity corresponds to the Stackelberg leader's capacity level. This explains the jump for the firms' capacities at $\hat{X}$. The increase of K_D^* for $X \geq \hat{X}$ is less dramatic than that for $X < \hat{X}$. This increase is only to reduce the follower's investment capacity rather than to postpone the follower's entry. Correspondingly to the increase in the leader's optimal capacity levels, the follower's optimal investment capacity decreases with X. More specifically, $K_F^*(X)$ decreases faster for $X < \hat{X}$ because of the over-investment effect and much slower for $X \geq \hat{X}$.

Figure 3.7 demonstrates the value for the dedicated leader $V_D^*(X)$ and the flexible follower $V_F^*(X)$. Note that if $X < X_D^{det}$, the value of the leader is the value of holding the option to invest, not the immediate value at the moment of investment as in Huisman and Kort (2015), thus it is not equal 0. If $X \geq X_D^{det}$, then the leader's value is the value of immediate investment. The leader switches from the entry deterrence to the entry accommodation strategy at $\hat{X}$. When $X < X_D^{det}$, the follower's value also comes from holding the option to invest. When $X_D^{det} \leq X < \hat{X}$, the follower expects the leader to play the entry deterrence strategy and to invest at X. Then the follower would adjust the investment timing

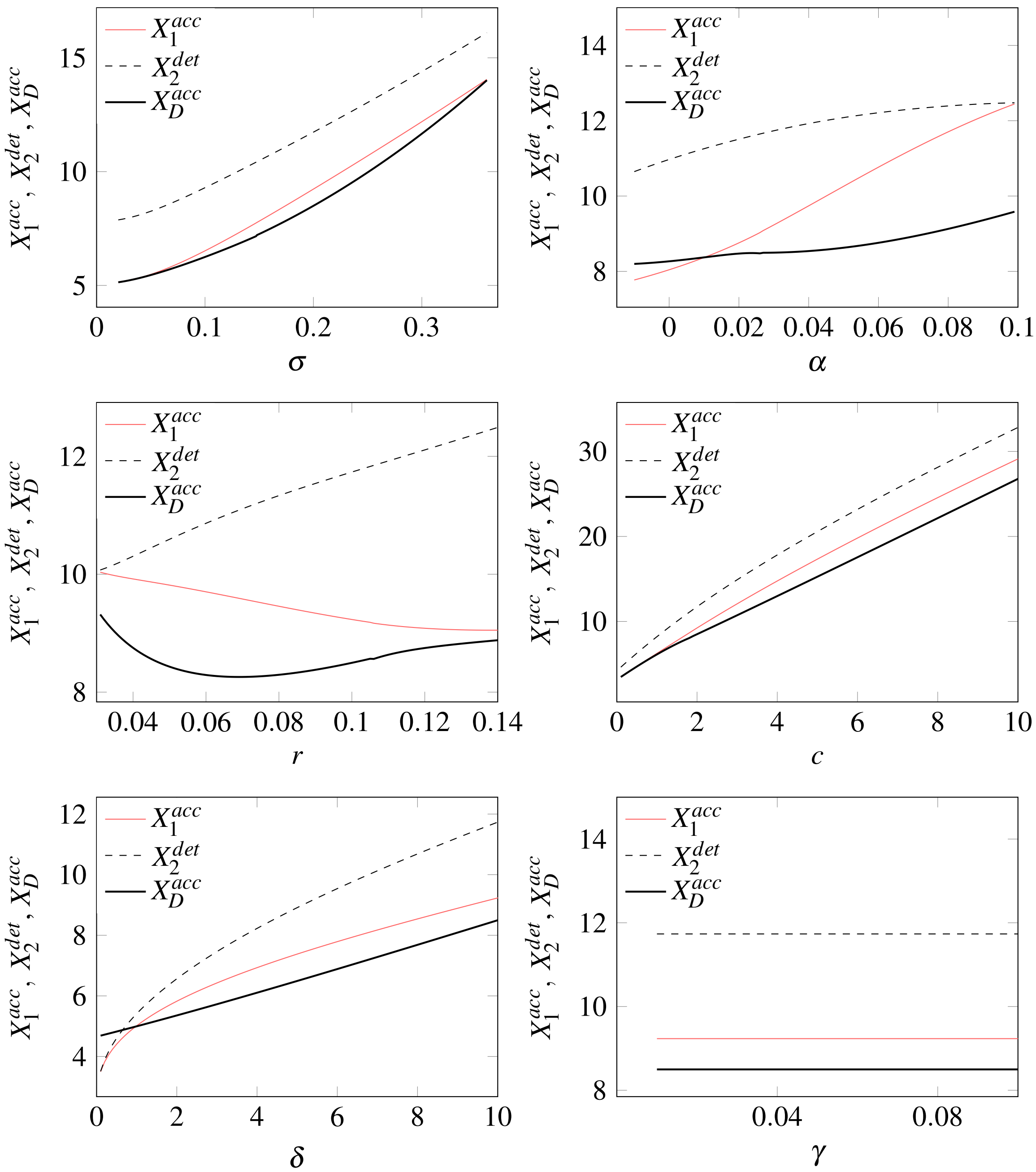


Figure 3.5: Illustration of X_1^{acc}, X_2^{det}, and X_D^{acc}. Default parameter values are $\alpha = 0.03$, $\sigma = 0.2$, $r = 0.1$, $c = 2$, $\delta = 10$, $\gamma = 0.05$.

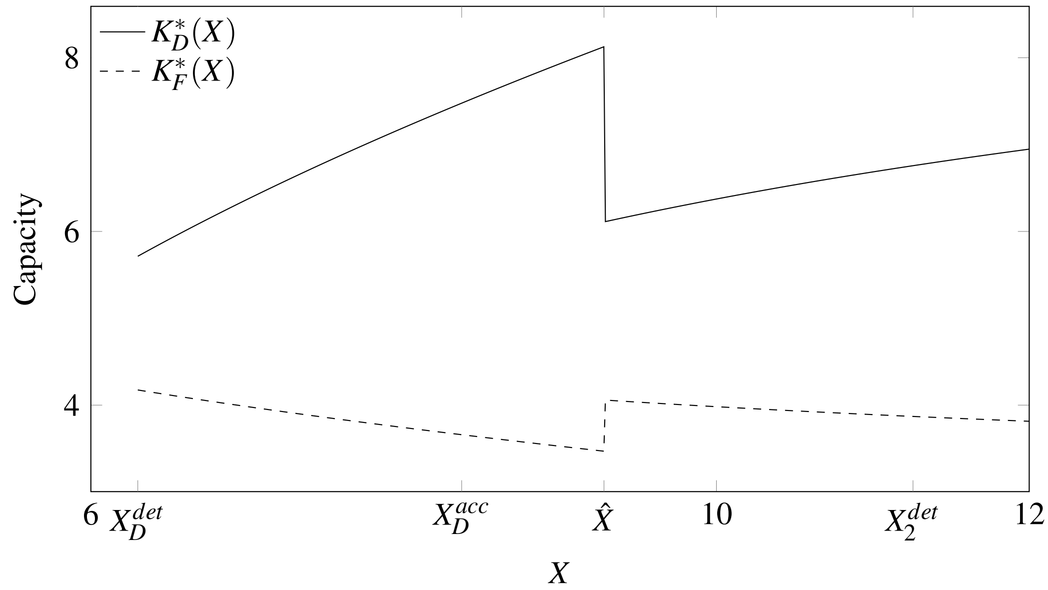


Figure 3.6: Illustration of $K_D^*(X)$ and $K_F^*(X)$ when the flexible follower produces up to capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.01$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

accordingly. For the given parameter values in Figure 3.7, the flexible follower does not invest for $X \leq \hat{X}$. It therefor, holds an option to invest until X rises above $\hat{X}$. The kink in the follower's value function is because the leader switches from entry deterrence to entry accommodation strategy, where the follower invests immediately at the same time with the leader.

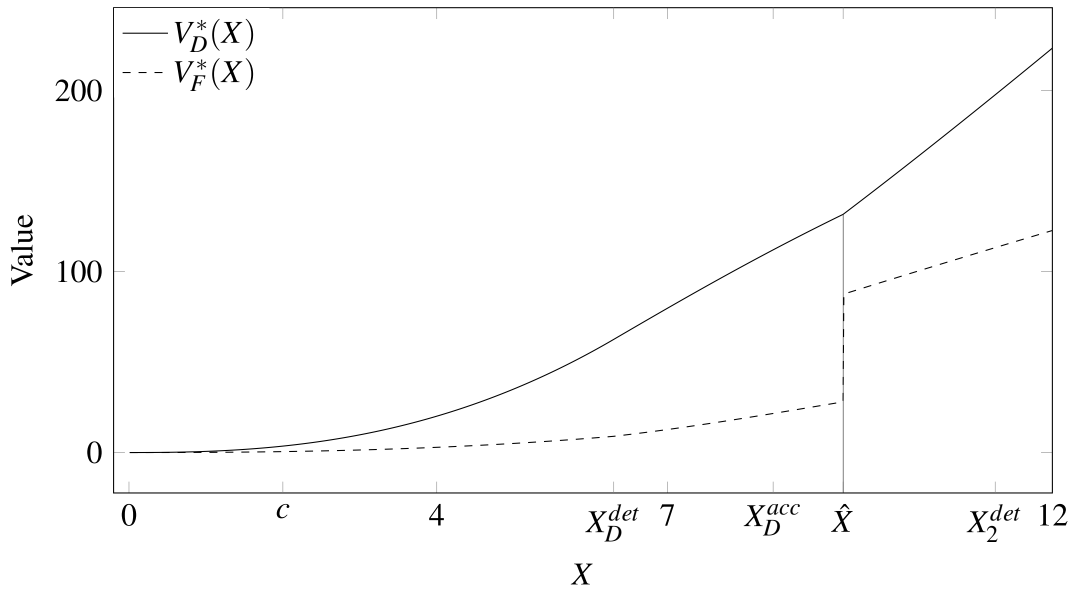


Figure 3.7: Illustration of $V_D^*(X)$ and $V_F^*(X)$ when the flexible follower produces up to capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.01$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

Figure 3.8 demonstrates the optimal capacity levels for the two firms when the follower produces below capacity right after investment. For given parameter values $r = 0.1$, $\alpha = 0.03$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, and $\delta = 0.5$, if $X < X_D^{det}$, the leader waits until X reaches $X_D^{det} = 4.3050$ to implement the entry deterrence strategy. The entry deterrence strategy will be chosen when $X < \hat{X} = 4.4779$. If $X \geq \hat{X}$, the leader chooses the entry accommodation strategy. However, different from the previous example where $X_D^{acc} < \hat{X}$, here $X_1^{acc} = 4.4072 < \hat{X} < X_D^{acc} = 4.8238$, implying that the leader chooses the accommodation strategy for $\hat{X} \leq X < X_D^{acc}$ but waits to invest until X_D^{acc} is reached. So the leader is holding an option to invest in the accommodation strategy. This is shown in Figure 3.8 as the void area when $\hat{X} \leq X < X_D^{acc}$.

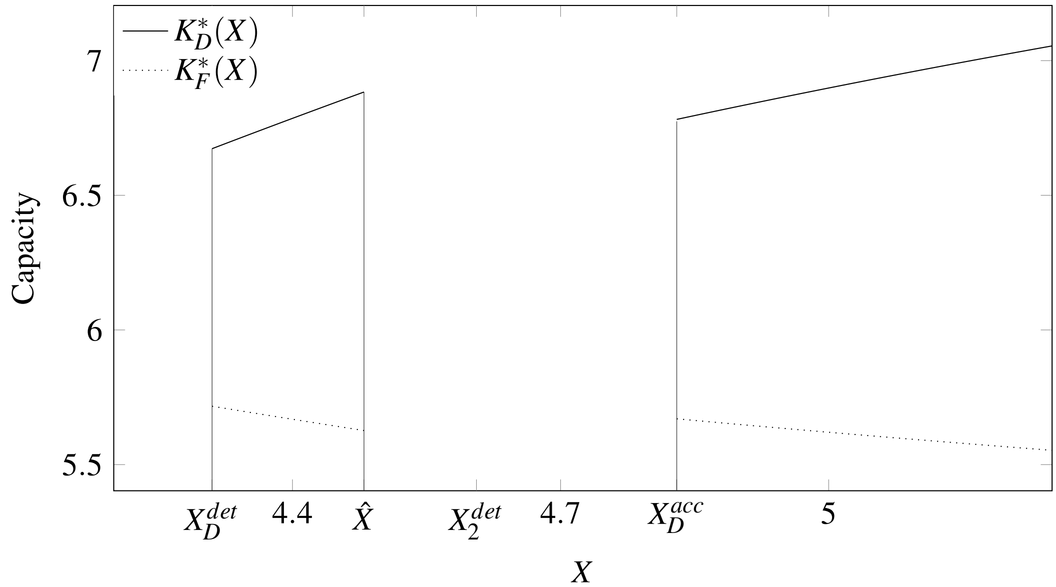


Figure 3.8: Illustration of $K_D^*(X)$ and $K_F^*(X)$ when the flexible follower produces below capacity right after investment. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\sigma = 0.2$, $\gamma = 0.05$, $c = 2$, $\delta = 0.5$.

Figure 3.9 demonstrates the values of the leader and the follower as functions of X when the follower produces below capacity right after investment. When $X < X_D^{det}$, the leader waits to invest with the entry deterrence strategy capacity. The follower is also waiting to invest, and expects the leader to invest at X_D^{det} with capacity $K_D^{det}(X_D^{det})$. For $X_D^{det} \leq X < \hat{X}$, the leader invests immediately at level X with deterrence capacity $K_D^{det}(X)$. The follower invests later than the leader. When $\hat{X} \leq X < X_D^{acc}$, the leader switches to entry accommodation strategy and waits to invest at X_D^{acc} with capacity $K_D^{acc}(X_D^{acc})$. The follower invests at the same time with the leader with capacity $K_F^*(K_D^{acc}(X_D^{acc}))$. Because of the switch of strategies, the leader's value function has a kink and the follower's value function is shown to jump at $\hat{X}$. When $X \geq X_D^{acc}$, the leader invests immediately with the entry accommodation strategy capacity $K_D^{acc}(X)$. The follower also invests at the same time as the leader with capacity $K_F^*(K_D^{acc}(X))$.

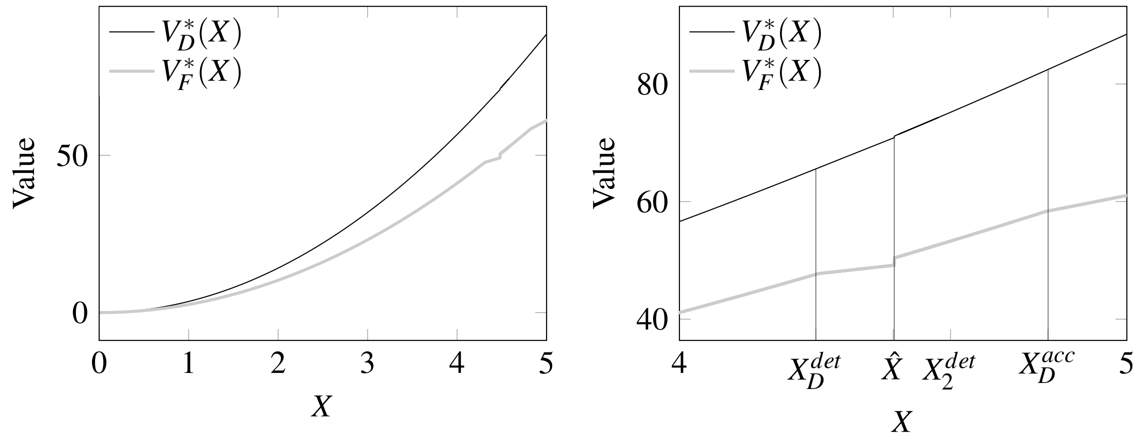


Figure 3.9: Illustration of $V_D^*(X)$ and $V_F^*(X)$ when the flexible follower produces below capacity right after investment. Parameter values are $\alpha = 0.03$, $\sigma = 0.2$, $r = 0.1$, $c = 2$, $\delta = 10$, $\gamma = 0.05$.

3.5 Influence of Flexibility

In order to analyze the influence of the follower's volume flexibility, the optimal investment decisions without flexibility are derived in the appendix. By comparing the leader's investment decisions with a flexible and with a dedicated follower, we get the following proposition.

Proposition 3.5. *Volume flexibility does not influence the leader's entry deterrence strategy. Moreover, it also does not influence the leader's optimal capacity under entry accommodation strategy.*

In this section, numerical analysis is carried out to investigate how flexibility influences the leader and follower's investment decisions. More specifically, it considers the possible occurrence of each strategy by comparing X_1^{det}, X_2^{det}, and X_1^{acc}. Because the flexibility influences the investment threshold for the accommodation strategy, the focus is put on the accommodation strategy, or rather on the switch between the entry deterrence and accommodation strategy. The impact of flexibility on the leader's accommodation strategy capacity and option values at this switch is also analyzed. Moreover, this section looks further at the follower's optimal investment decisions under leader's entry deterrence and accommodation strategies. The follower's investment thresholds and capacities are compared according to whether the production flexibility is available. The influence of the flexibility on the follower's values at the moment of investment is also analyzed.

3.5.1 Flexibility Influences Dedicated Leader

Though the follower's flexibility influences neither the leader's deterrence strategy, nor the investment capacity under the accommodation strategy, it does influence the possibility to implement these two strategies. The analysis is focused on the interval $[X_1^{det}, X_2^{det}]$, where the entry deterrence strategy is considered; and the region in which $X \geq X_1^{acc}$, where the accommodation strategy is considered.

Figure 3.10 demonstrates that, under entry deterrence strategy, X_1^{det} with flexibility is greater or equal to that without flexibility and X_2^{det} with flexibility is smaller or equal to that without flexibility. Thus the interval to implement entry deterrence strategy shrinks when the follower is flexible. For the entry accommodation strategy, X_1^{acc} with flexibility is smaller than that without flexibility, so the interval to implement entry accommodation strategy enlarges when the follower is flexible. The changes in the intervals reflect the tendency for the leader to implement the corresponding strategy. It holds that for the given parameters, the leader tends to delay the flexible follower's entry less and is more likely to choose the accommodation strategy in case of a flexible follower.

Figure 3.11 shows that the dedicated leader's accommodation strategy threshold is higher than that without flexibility, because a flexible follower invests more than an inflexible follower at X_D^{acc} and a higher demand is required to choose the accommodation strategy. For the given parameter values, we have $X_1^{acc} > X_D^{acc}$, which makes the optimal threshold X_D^{acc} meaningless as in the case of without flexibility. From Proposition 3.4, the leader does not necessarily invest at threshold X_D^{acc} to apply the entry accommodation. If $\hat{X} \geq X_D^{acc}$, the leader invests at $\hat{X}$, the switch from entry deterrence to accommodation. So for the accommodation strategy, we further analyze the influence of the follower's flexibility on $\hat{X}$. In Figure 3.11, it is shown that $\hat{X} > X_1^{acc}$, implying that $\hat{X}$ is meaningful. Moreover, $\hat{X}$ increases with market uncertainty σ. This means that the leader switches to the accommodation strategy later in a more volatile market. The intuition is that both the leader and the follower invest more in case of a larger uncertainty, see also Figure 3.15. Therefore, a higher demand is required to accommodate the two firms. However, $\hat{X}$ with flexibility is smaller than without flexibility, implying that the dedicated leader switches to the accommodation strategy earlier when the follower is flexible. This is similar as the findings above that the accommodation strategy is more likely with a flexible follower. Furthermore, Figure 3.11 also demonstrates that when switching to the accommodation strategy, the leader invests less if the follower is flexible. This will be explained in the following subsection.

Next, we check how the follower's flexibility affects the leader's value. Figure 3.12 demonstrates the dedicated leader's value under entry deterrence strategy at investment threshold X_D^{det} and under accommodation strategy at $\hat{X}$, with and without flexibility. For the entry deterrence strategy, Figure 3.12a shows that the leader's value at the investment threshold X_D^{det} with flexibility is smaller or equal to that without flexibility. This is because

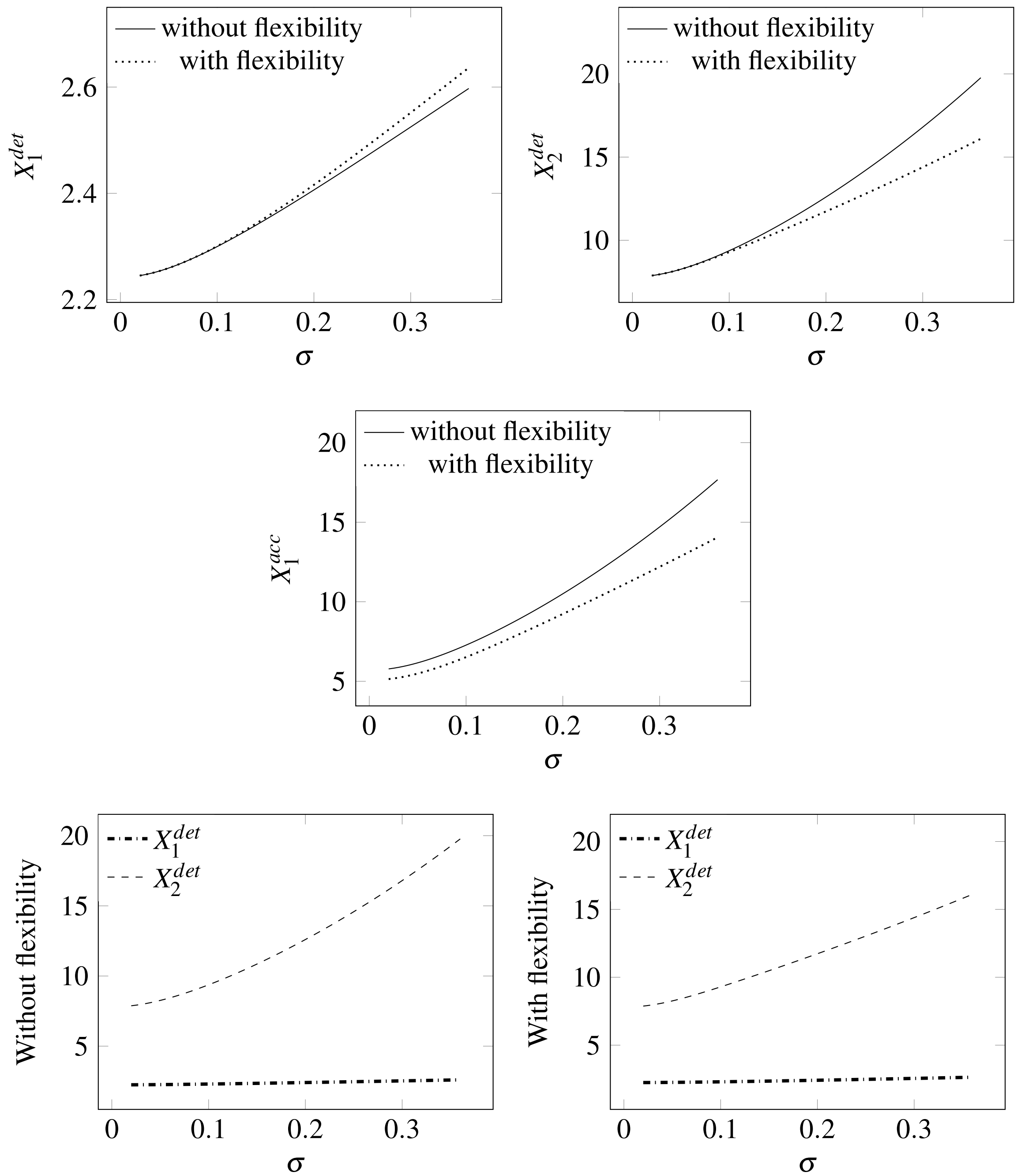


Figure 3.10: Illustration of X_1^{det}, X_2^{det}, and X_1^{acc} with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

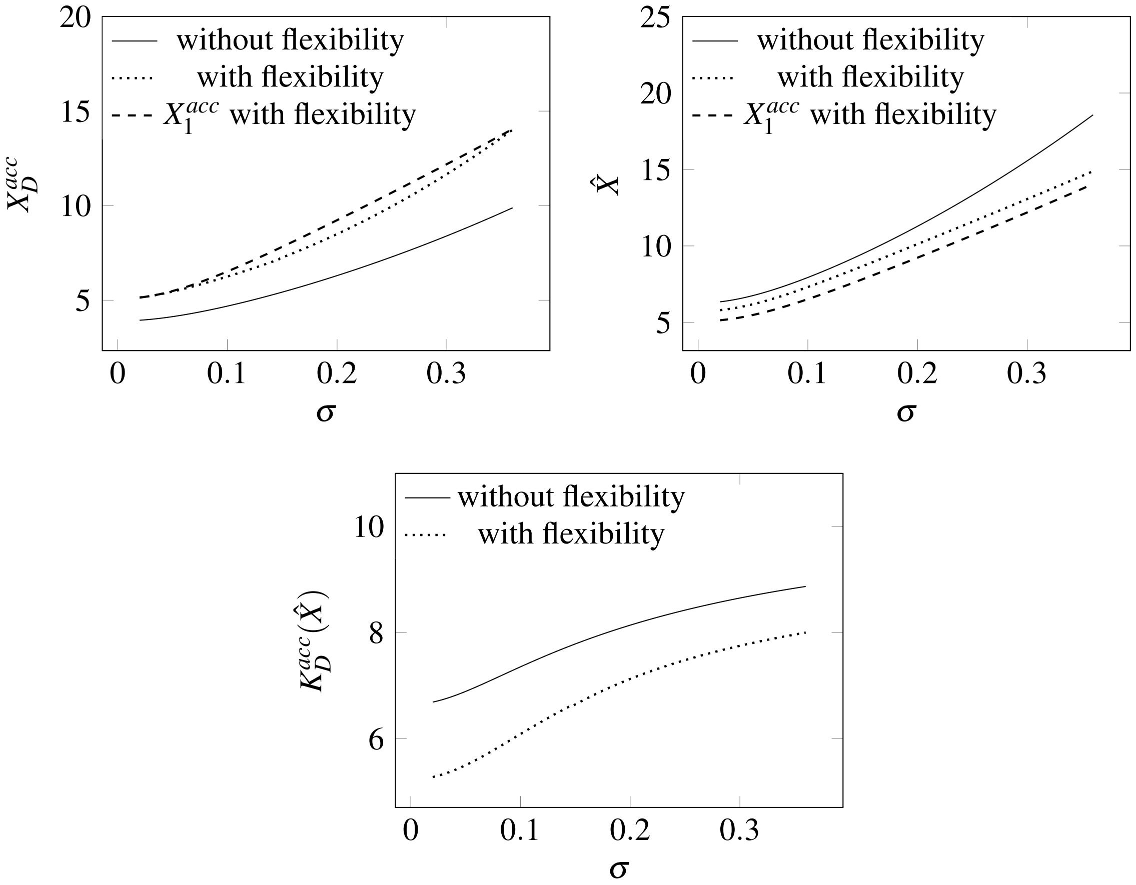
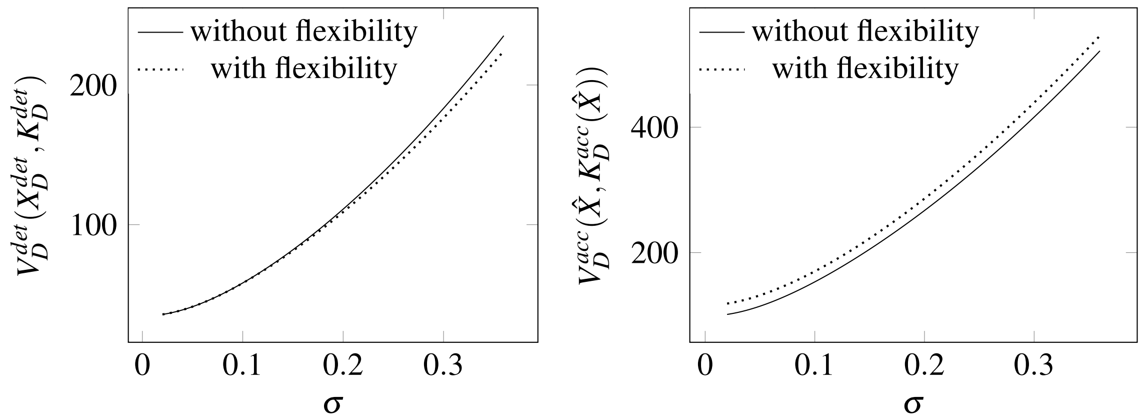


Figure 3.11: Illustration of X_D^{acc}, $\hat{X}$, and $K_D^{acc}(\hat{X})$ with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

flexibility does not change the dedicated leader's investment threshold and capacity under entry deterrence strategy, but it makes the follower enter the market earlier (see Figure 3.13), which puts an earlier end to the leader's monopoly privilege. Similar analysis can be done for the accommodation strategy. The follower's flexibility makes the leader invest earlier and less under accommodation strategy. However, as shown in Figure 3.12, the leader's values at the moment of investment is larger than that without flexibility under the accommodation strategy[4]. This implies that the follower's flexibility also is good for the leader when accommodating the flexible follower's entry. If the leader deters the flexible follower's entry, the flexibility decreases its value.



(a) Entry deterrence strategy. (b) Deterrence and accommodation switch.

Figure 3.12: Illustration of $V_D^{det}(X_D^{det}, K_D^{det})$ when investing at the optimal threshold X_D^{det}, and $V_D^{acc}(\hat{X}, K_D^{acc}(\hat{X}))$ when investing at level $\hat{X}$ with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

3.5.2 Flexibility Influences Flexible Follower

In this subsection, we compare how the flexibility influences the follower's investment threshold, capacity, and value when the dedicated leader takes the entry deterrence and accommodation strategies.

When the leader chooses the entry deterrence strategy, the flexibility allows a flexible follower to invest with more capacity, as shown in Figure 3.13. This is because the follower can adjust output levels according to market demand, and invests more anticipating potential upward demand shocks in the future. According to the analysis of the monopoly case as in Wen et al. (2017), larger capacity means more investment costs, and the firm invests later

[4]Note that $\hat{X}$ is different for the leader depending on the follower's flexibility. When comparing values at different $\hat{X}$s, the comparison should be made at a predetermined point of time, for instance, at $\hat{X}$ in the flexible follower situation. The discount factor for the leader's value with a flexible follower is $\left(\hat{X}_{inflexible}/\hat{X}_{flexible}\right)^{\beta_1}$, where $\hat{X}_{flexible}$ stands for the $\hat{X}$ in the flexible follower situation, and $\hat{X}_{inflexible}$ stands for the $\hat{X}$ in the inflexible follower situation.

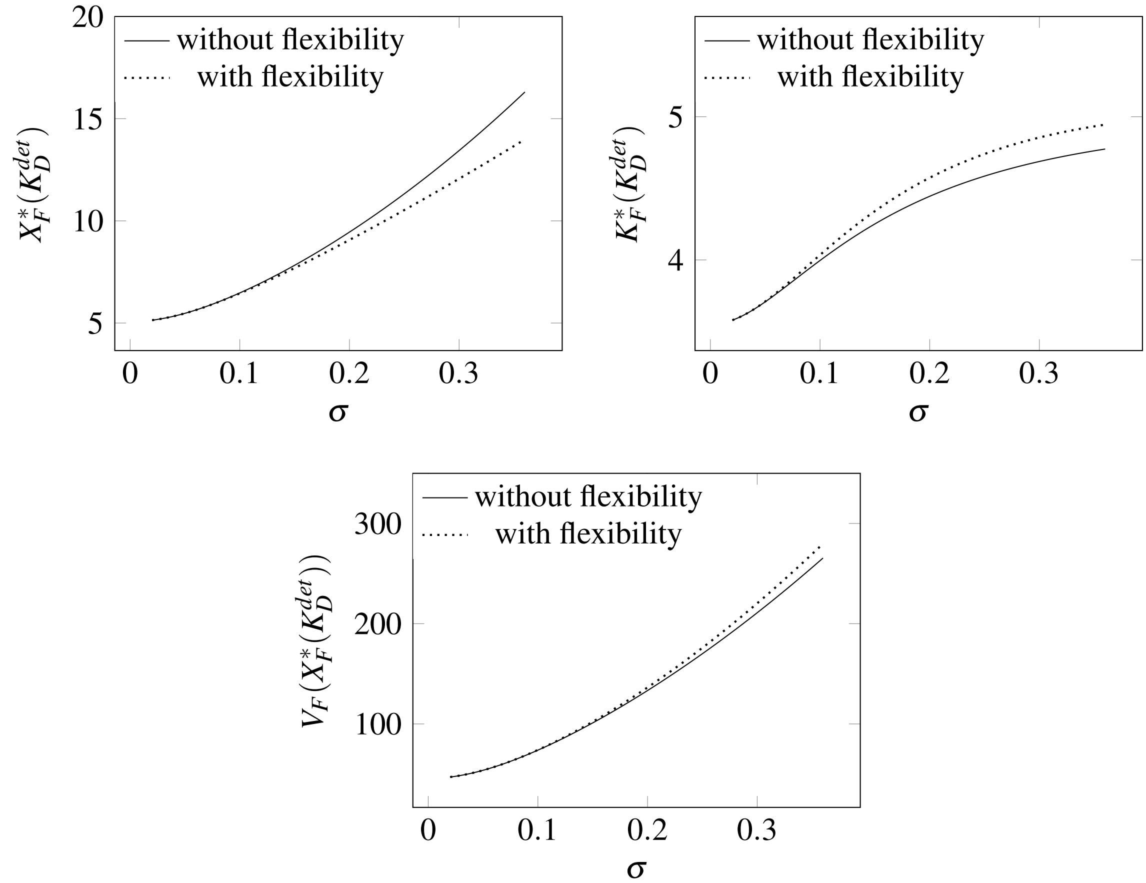


Figure 3.13: Illustration of $X_F^*(K_D^{det})$, $K_F^*(K_D^{det})$, and $V_F(X_F^*(K_D^{det}))$ under entry deterrence strategy with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

so that the market demand is higher to compensate for the investment costs. However, as shown in Figure 3.13, this is not the case in the duopoly model. Here, the flexible follower invests more and invests earlier than the inflexible follower. Twofold reasons are provided. On one hand, the inflexible follower always produces up to full capacity, even when the profit is negative for low levels of X. This decreases the optimal investment capacity, and delays inflexible follower's investment because of the preference for large market demand. On the other hand, the flexible follower can invest more because the future output quantity can adjust to market demand. The flexibility also gives higher values to the follower as illustrated in Figure 3.13. The higher value is due to the flexibility to avoid overproduction in case of low demands, and motivates the follower to invest earlier. Besides, the difference between with and without flexibility increases with σ. This is because for smaller σ, market uncertainty is low and the flexible follower produces up to capacity right after investment, so the difference in $X_F^*(K_D^{det})$ and $K_F^*(K_D^{det})$ between with and without flexibility is relatively small. However, with more market uncertainty, i.e., larger σ, the flexible follower produces below capacity right after investment and more capacity is put on hold for future positive demand shocks, so the difference is relatively large.

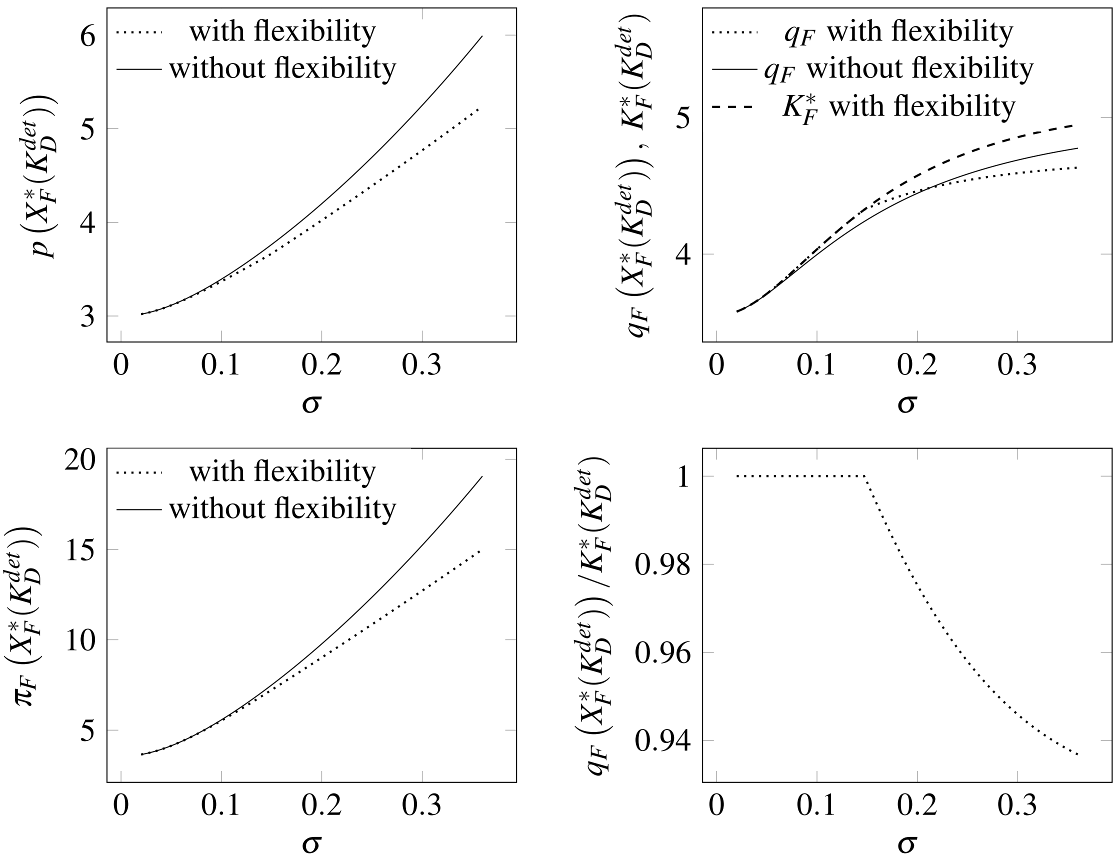


Figure 3.14: Illustration of $p\left(X_F^*(K_D^{det})\right)$, $q_F\left(X_F^*(K_D^{det})\right)$, $\pi(X_F^*(K_D^{det}))$, and $q_F\left(X_F^*(K_D^{det})\right)/K_F^*(K_D^{det})$ with and without flexibility under the entry deterrence strategy. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

We can also compare at the moment of the follower's investment the market price $p(X_F^*(K_D^{det}))$, the follower's output levels $q_F\left(X_F^*(K_D^{det})\right)$, the profit $\pi(X_F^*(K_D^{det}))$, and the utilization rate $q_F\left(X_F^*(K_D^{det})\right)/K_F^*(K_D^{det})$, i.e., how much of the capacity is used for production . As demonstrated in Figure 3.14, the market price is higher when the inflexible follower invests. The market price is influenced by both the leader and follower's output levels and the market demand. Figure 3.14 shows that when the flexible follower produces up to capacity for small σ, the output is close to or slightly larger than that of the inflexible follower at the moment of investment. Because the inflexible follower invests later, the market demand is larger. These two effects lead to higher instant market prices for the inflexible follower when it invests, given the same entry deterrence strategy of the leader. When the flexible follower produces below capacity, the flexible follower's instant output gradually falls below the inflexible follower's output as σ increases. Given the same output levels from the leader, there is less instant output from the follower. However, the inflexible follower invests much later than the flexible follower, so the market demand is much higher. Thus, the instant market price when the inflexible follower invests is still higher. This also leads to higher instant profit flows of the inflexible follower. However, the flexible follower invests earlier than the inflexible follower, if the flexible follower's value is "discounted" to the inflexible follower's investment threshold, as shown in Figure 3.13, the flexible follower has higher values[5]. Figure 3.14 also demonstrates that the utilization rate at the moment of investment is 1 when the flexible follower produces up to capacity, and it decreases with σ when the flexible follower produces below capacity. The latter result is consistent with the findings in Hagspiel et al. (2016). The reason is that a higher σ implies larger market uncertainty, and for a positive market trend α, the firm invests later with more capacity as shown in Figure 3.13. Although for the output decisions, only current market demand matters. The investment being delayed implies larger market demand, and the output also increases with σ. However, as shown in Figure 3.14, the invested capacity $K_F^*(K_D^{det})$ increases faster then $q_F(X_F^*(K_D^{det}))$, thus the utilization rate decreases with σ.

The dedicated leader switches from deterrence to accommodation strategy at $\hat{X}$. Note that for the accommodation strategy, the follower invests at the same time with the leader, thus in Figure 3.15 $X_F^*(K_D^{acc}(\hat{X}))$ is the same as $\hat{X}$ in Figure 3.11. Figure 3.15 shows that under the leader's accommodation strategy, the flexible follower invests earlier and more than the inflexible follower. The story here is similar to that in the deterrence strategy, where the flexible follower also invests earlier and more than the inflexible follower, and has higher value.

Figure 3.16 demonstrates the market price, the follower's output levels, profit flows and flexible follower's utilization rate at the moment of investment when the leader takes entry accommodation strategy. As shown in Figure 3.16 and unlike that for the entry

[5]The discount factor is $\left(X^*_{F\ inflexible}/X^*_{F\ inflexible}\right)^{\beta_1}$, where $X^*_{F\ inflexible}$ represents $X_F^*(K_D^{det})$ for the inflexible follower and $X^*_{F\ flexible}$ represents $X_F^*(K_D^{det})$ for the flexible follower.

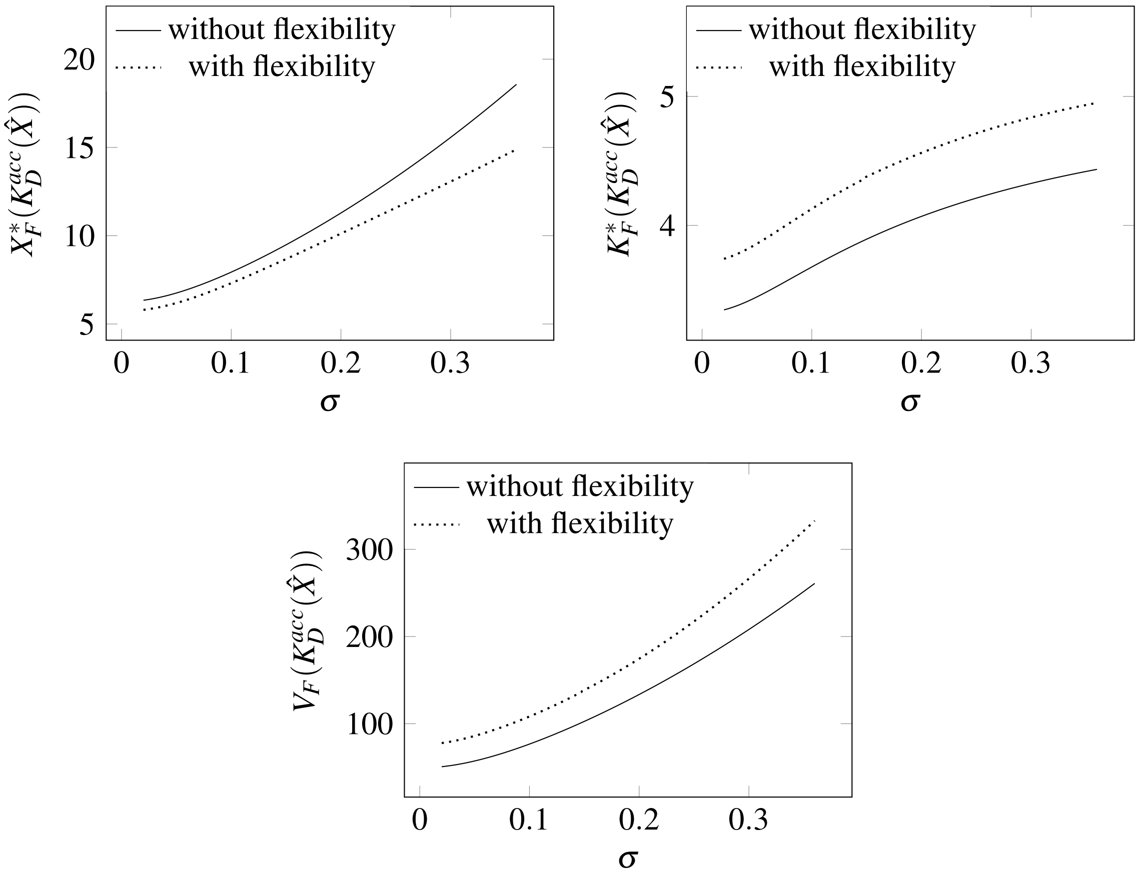


Figure 3.15: Illustration of $X_F^*(K_D^{acc}(\hat{X}))$, $K_F^*(K_D^{acc}(\hat{X}))$, and $V_F(K_D^{acc}(\hat{X}))$ under the entry accommodation strategy with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

deterrence strategy, the market price at the moment of investment when the follower is flexible, is not always lower than when the follower is inflexible. There are three factors that affect market prices: the market demand at the moment of investment, the follower's output quantity, and the leader's installed capacity. Because the flexible follower invests earlier as shown in Figure 3.15, the market demand is smaller. The output $q_F(K_D^{acc}(X))$ in Figure 3.16 shows that the flexible follower produces more than an inflexible follower. Thus, if the leader produces the same output level as in the entry deterrence strategy, the market price with a flexible follower would be lower than that with an inflexible follower. However, Figure 3.11 shows that the leader installs smaller capacity when the follower is flexible. Because the leader always produces up to capacity, this might result in a market price higher than that with an inflexible follower. For smaller σ, the follower's profit at the moment of investment, $\pi_F(K_D^{acc}(\hat{X}))$ is larger when the follower is flexible. Whereas for larger σ, this profit is larger when the follower is inflexible. The reasons are as follows. For smaller σ, the price levels at the moment of investment are higher with flexibility, and the flexible follower is producing more than the inflexible follower. Thus the flexible follower has larger profit at the moment of investment. However, when σ is large, the price level at the moment of investment is much higher for the inflexible follower. Though the flexible follower still produces more than an inflexible follower, the later has higher profit flows at the moment of investment. Figure 3.16 also demonstrates that when the flexible follower produces below capacity right after investment, the utilization rate decreases with σ. The intuition is the same as for the deterrence strategy case.

3.5.3 First Mover Advantage v.s. Technological Advantage

In this subsection, we investigate whether the leader's first mover advantage can be overcome by the flexible follower's technological advantage.

Figure 3.17 compares the leader and the follower's values for the leader's entry deterrence and accommodation strategy with and without flexibility. The leader always has higher values than the follower, implying the first mover advantage cannot be leapfrogged by the volume flexibility advantage. For the duopoly model in this chapter, according to Corollary 3.1, the optimal installed capacity by the follower decreases with the leader's installed capacity size. So this result is supported by Gal-Or (1985). They find that the leader has higher payoff than the follower if the reaction function of the follower is downward-sloping. The players are symmetric in Gal-Or's model. In our model, the players are asymmetric and time is continuous. The possible reason, why it is more difficult for the technological advantage to take over the first mover advantage, is that there are benefits to the leader by the follower's production technology without sharing costs.

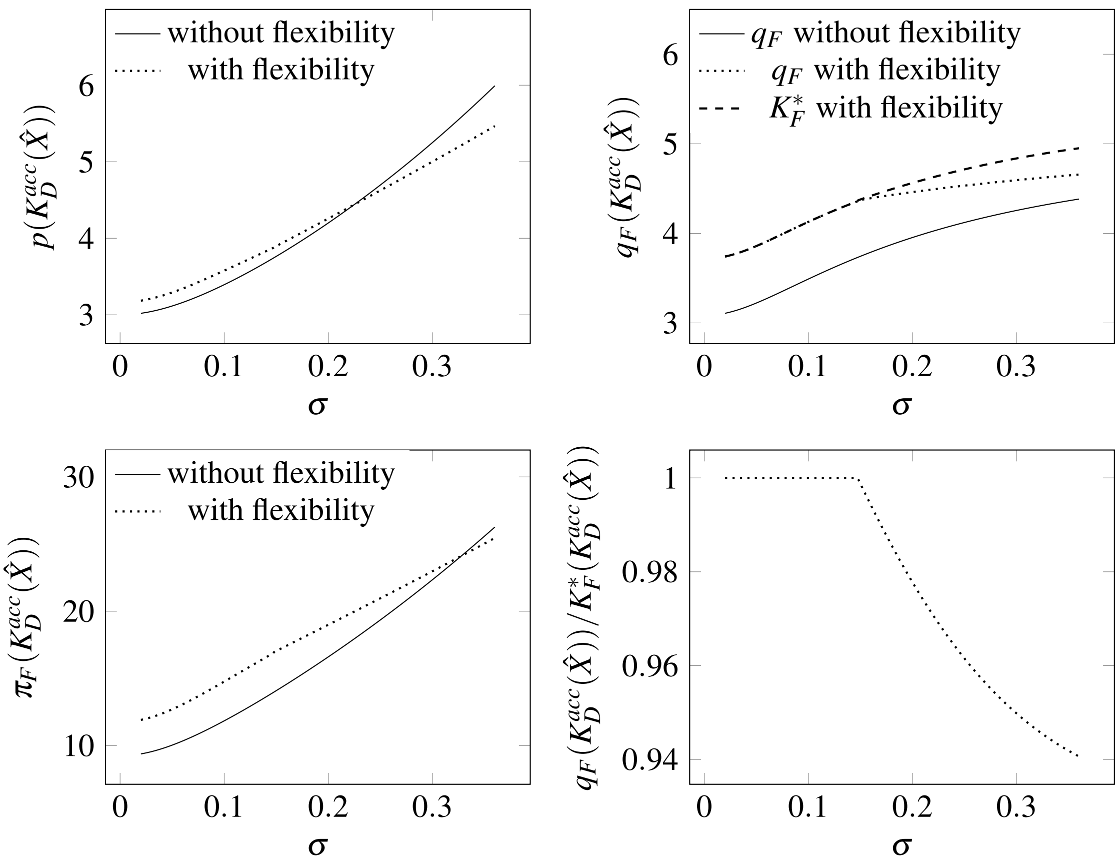


Figure 3.16: Illustration of $p(K_D^{acc}(\hat{X}))$, $q_F(K_D^{acc}(\hat{X}))$, $\pi_F(K_D^{acc}(\hat{X}))$, and $q_F(K_D^{acc}(\hat{X}))/K_F^*(K_D^{acc}(\hat{X}))$ with and without flexibility under the entry accommodation strategy. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

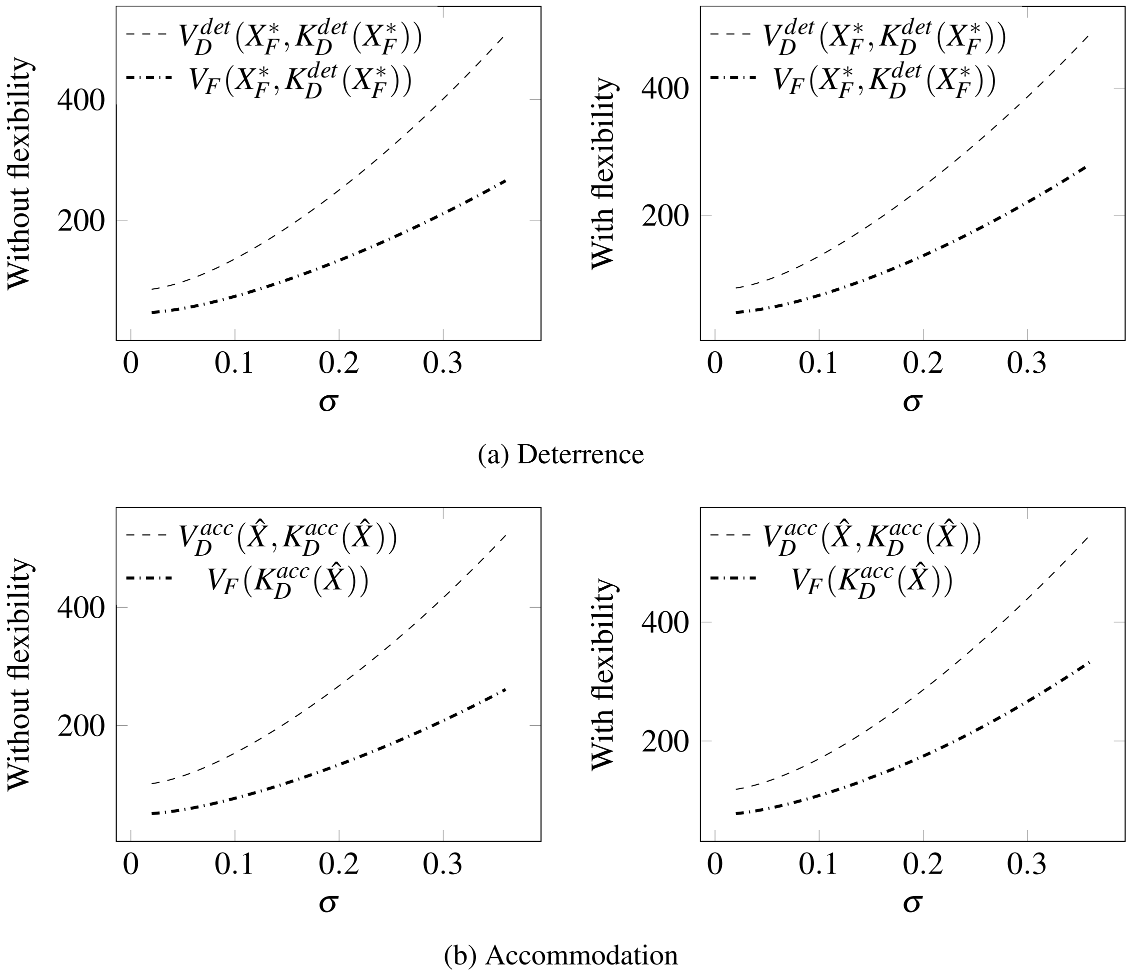


Figure 3.17: Comparison of $V_D^{det}(X_F^*, K_D^{det}(X_F^*))$ and $V_F(X_F^*, K_D^{det}(X_F^*))$ under the entry deterrence strategy, $V_D^{acc}(\hat{X}, K_D^{acc}(\hat{X}))$ and $V_F(K_D^{acc}(\hat{X}))$ under the entry accommodation strategy, with and without flexibility. Parameter values are $r = 0.1$, $\alpha = 0.03$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

3.6 Conclusion

This chapter introduces volume flexibility into the strategic capacity investment problem under uncertainty. In the duopoly framework, the follower has technological advantage over the leader in that the follower can adjust output quantity within the constraint of installed production capacity, and the leader always produces up to capacity. When making decisions about investment timing and investment capacity, the leader not only takes into account the incentives to preempt, but also the influence of the follower's volume flexibility on the market price. This is because the flexible follower competes against the dedicated leader on one hand, and on the other hand makes the market price fluctuate less when there is demand uncertainty. We show that compared with a dedicated follower, the dedicated leader is more likely to accommodate the entry of the flexible follower. This is due to the fact that entry deterrence strategy decreases the leader's value when the follower is flexible, and the entry accommodation strategy increases the leader's value. The leader does not like to play entry deterrence because volume flexibility enables the follower to enter the market earlier and thus shortens the leader's monopoly period. Whereas when playing the accommodation strategy, two firms enter the market later than that under the deterrence strategy, so the market demand is larger. In a way, the leader benefits more from the less fluctuating market prices due to follower's volume flexibility. Dixit (1980) proves that in a static setting, entry deterrence is largely ineffective if the leader cannot commit to producing at full capacity, because the leader facing irrevocable entry finds it optimal to make an accommodating output reduction. We prove that in a stochastic dynamic setting, deterring the entry of a flexible follower does not necessarily make the leader better off, and the leader's commitment to an output quantity is still ineffective to deter the follower's entry. In fact, the leader commits to the same output level under the deterrence and accommodation strategy, but invests at different timings. The establishment of the role for uncertainty is also an attempt to answer to Huisman and Kort (2015).

Our model assumes exogenous firm roles and takes the dedicated firm as the leader and the flexible firm as the follower in the market. We have shown that technology advancement cannot overtake the first mover advantage in the sense that the dedicated leader has higher value than the flexible follower. So it might be interesting to see how asymmetric firms interact with each other strategically in an endogenous firm role setting. Another possible extension is to investigate other demand structures. Our model assumes multiplicative demand function. We find the dedicated leader installs monopolistic capacity size for the entry deterrence and accommodation strategy. This is a strong result. It is worthwhile to do a robustness check with a different demand function.

3.7 Appendix

3.7.1 Derivations and Proofs

Derivation of $L_1(K_D,K_F), M_1(K_D,K_F), M_2(K_D)$**,** $N_2(K_D,K_F)$ Let $X_1 = c/(1-\gamma K_D)$ and $X_2 = c/(1-\gamma K_D - 2\gamma K_F)$. Employing value matching and smooth pasting conditions gives the following equations:

$$
\begin{aligned}
L(K_D,K_F) =& M_1(K_D,K_F) + \frac{X_1^{-\beta_1}}{4\gamma(\beta_1-\beta_2)}\left[\frac{2c\beta_2(1-\gamma K_D)}{r}\right.\\
&\left.-\frac{X_1(1-\gamma K_D)^2(\beta_2-1)}{r-\alpha} - \frac{c^2(\beta_2+1)}{(r+\alpha-\sigma^2)X_1}\right]\\
=&\frac{c^{1-\beta_1}(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)}\left[-\frac{2\beta_2}{r}+\frac{\beta_2-1}{r-\alpha}+\frac{\beta_2+1}{r+\alpha-\sigma^2}\right]\\
&+\frac{c^{1-\beta_1}(1-\gamma K_D)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)}\left[\frac{2\beta_2}{r}-\frac{\beta_2-1}{r-\alpha}-\frac{1+\beta_2}{r+\alpha-\sigma^2}\right]\\
=&\frac{(1-\gamma K_D)^{1+\beta_1}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)c^{\beta_1-1}}\left[\frac{2\beta_2}{r}-\frac{\beta_2-1}{r-\alpha}-\frac{1+\beta_2}{r+\alpha-\sigma^2}\right],\\
M_1(K_D,K_F) =& \frac{X_2^{-\beta_1}}{\beta_1-\beta_2}\left\{\frac{cK_F\beta_2}{r}+\frac{(1-\beta_2)(1-\gamma K_D-\gamma K_F)K_FX_2}{r-\alpha}\right.\\
&\left.\frac{1}{4\gamma}\left[-\frac{2c\beta_2(1-\gamma K_D)}{r}+\frac{X_2(\beta_2-1)(1-\gamma K_D)^2}{r-\alpha}+\frac{c^2(\beta_2+1)}{(r+\alpha-\sigma^2)X_2}\right]\right\}\\
=&\frac{X_2^{-\beta_1}(1-2\gamma K_F-\gamma K_D)}{4\gamma(\beta_1-\beta_2)}\left[-\frac{2c\beta_2}{r}+\frac{(\beta_2-1)c}{r-\alpha}+\frac{c(\beta_2+1)}{r+\alpha-\sigma^2}\right]\\
=&\frac{c^{1-\beta_1}(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)}\left[-\frac{2\beta_2}{r}+\frac{\beta_2-1}{r-\alpha}+\frac{\beta_2+1}{r+\alpha-\sigma^2}\right],\\
M_2(K_D) =& \frac{X_1^{-\beta_2}}{4\gamma(\beta_1-\beta_2)}\left[\frac{2c\beta_1(1-\gamma K_D)}{r}-\frac{X_1(\beta_1-1)(1-\gamma K_D)^2}{r-\alpha}-\frac{c^2(\beta_1+1)}{(r+\alpha-\sigma^2)X_1}\right]\\
=&\frac{c^{1-\beta_2}(1-\gamma K_D)^{1+\beta_2}}{4\gamma(\beta_1-\beta_2)}\left[\frac{2\beta_1}{r}-\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1+1}{r+\alpha-\sigma^2}\right],\\
N(K_D,K_F) =& M_2(K_D) + \frac{X_2^{-\beta_2}}{\beta_1-\beta_2}\left\{\frac{1}{4\gamma}\left[-\frac{2c\beta_1(1-\gamma K_D)}{r}+\frac{X_2(\beta_1-1)(1-\gamma K_D)^2}{r-\alpha}\right.\right.\\
&\left.\left.+\frac{c^2(\beta_1+1)}{(r+\alpha-\sigma^2)X_2}\right]+\frac{cK_F\beta_1}{r}+\frac{(1-\beta_1)(1-\gamma K_D-\gamma K_F)K_FX_2}{r-\alpha}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= M_2(K_D) + \frac{X_2^{-\beta_2}(1-\gamma K_D - 2\gamma K_F)}{4\gamma(\beta_1-\beta_2)}\left[-\frac{2c\beta_1}{r} + \frac{c(\beta_1-1)}{r-\alpha} + \frac{c(\beta_1+1)}{r+\alpha-\sigma^2}\right] \\
&= \frac{(1-\gamma K_D)^{1+\beta_2} - (1-\gamma K_D - 2\gamma K_F)^{1+\beta_2}}{4\gamma(\beta_1-\beta_2)c^{\beta_2-1}}\left[\frac{2\beta_1}{r} - \frac{\beta_1-1}{r-\alpha} - \frac{1+\beta_1}{r+\alpha-\sigma^2}\right].
\end{aligned}
$$

Let

$$
\begin{aligned}
F(\beta) &= \frac{2\beta}{r} - \frac{\beta-1}{r-\alpha} - \frac{\beta+1}{r+\alpha-\sigma^2} \\
&= \frac{\beta\left(2\alpha\sigma^2 - r\sigma^2 - 2\alpha^2\right) + r\left(2\alpha-\sigma^2\right)}{r(r-\alpha)(r+\alpha-\sigma^2)}.
\end{aligned}
$$

From $r > \alpha$, $r > \sigma^2 - \alpha$, it follows from Wen et al. (2017) that $\beta_1 > 1$, $\beta_2 < -1$, $F(\beta_1) < 0$, and $F(\beta_2) > 0$. Thus, we conclude that $L(K_D, K_F) > 0$, $M_1(K_D, K_F) < 0$, $N(K_D, K_F) > 0$, and $M_2(K_D) < 0$ for the duopoly model.

Proof of Proposition 3.1 The optimal investment capacity $K_F(X, K_D)$ of the follower maximizes $V_F(X, K_D, K_F) - \delta K_F$. The analysis is carried out for three different regions.

- Region 1: $0 < X < c/(1-\gamma K_D)$.

  In this region, we have

$$
\begin{aligned}
&V_F(X, K_D, K) - \delta K_F = L_1(K_D, K_F)X^{\beta_1} - \delta K_F \\
=& \frac{c^{1-\beta_1}\left[(1-\gamma K_D)^{1+\beta_1} - (1-2\gamma K_F - \gamma K_D)^{1+\beta_1}\right]F(\beta_2)}{4\gamma(\beta_1-\beta_2)}X^{\beta_1} - \delta K_F.
\end{aligned}
$$

  Taking the first order condition with respect to K_F gives

$$
\frac{c^{1-\beta_1}(1+\beta_1)F(\beta_2)X^{\beta_1}}{2(\beta_1-\beta_2)}(1-2\gamma K_F - \gamma K_D)^{\beta_1} - \delta = 0. \tag{3.35}
$$

  Thus,

$$
K_F(X, K_D) = \frac{1}{2\gamma}\left\{1 - \gamma K_D - \frac{c}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{cF(\beta_2)(1+\beta_1)}\right]^{\frac{1}{\beta_1}}\right\}. \tag{3.36}
$$

  The second order partial derivative of $V_F(X, K_D, K_F) - \delta K_F$ with respect to K_F is

$$
-\frac{\beta_1\gamma F(\beta_2)(1+\beta_1)}{\beta_1-\beta_2}\left(\frac{1-2\gamma K_F - \gamma K_D}{c}\right)^{\beta_1-1}X^{\beta_1} < 0.
$$

Thus, $K_F(X,K_D)$ maximizes $V_F(X,K_F,K_D)-\delta K_F$.

- Region 2: $c/(1-\gamma K_D) \le X < c/(1-\gamma K_D - 2\gamma K_D)$.

  In this region, we have

$$
\begin{aligned}
& V_F\left(X,K_D,K_F\right)-\delta K_F \\
=& M_1\left(K_D,K_F\right)X^{\beta_1}+M_2\left(K_D\right)X^{\beta_2}+\frac{\left(1-\gamma K_D\right)^2 X}{4\gamma(r-\alpha)}-\frac{c\left(1-\gamma K_D\right)}{2\gamma r} \\
& +\frac{c^2}{4\gamma X\left(r+\alpha-\sigma^2\right)}-\delta K_F \\
=& -\frac{c^{1-\beta_1}\left(1-2\gamma K_F-\gamma K_D\right)^{1+\beta_1}}{4\gamma(\beta_1-\beta_2)}F\left(\beta_2\right)X^{\beta_1}+\frac{c^{1-\beta_2}\left(1-\gamma K_D\right)^{1+\beta_2}}{4\gamma(\beta_1-\beta_2)}F\left(\beta_1\right)X^{\beta_2} \\
& +\frac{\left(1-\gamma K_D\right)^2 X}{4\gamma(r-\alpha)}-\frac{c\left(1-\gamma K_D\right)}{2\gamma r}+\frac{c^2}{4\gamma X\left(r+\alpha-\sigma^2\right)}-\delta K_F .
\end{aligned}
$$

  Taking the first order condition with respect to K_F gives

$$
K_F\left(X,K_D\right)=\frac{1}{2\gamma}\left\{1-\gamma K_D-\frac{c}{X}\left[\frac{2\delta\left(\beta_1-\beta_2\right)}{cF\left(\beta_2\right)\left(1+\beta_1\right)}\right]^{\frac{1}{\beta_1}}\right\}. \tag{3.37}
$$

  The second order partial derivative of with respect to K_F is

$$
-\frac{\beta_1\gamma F(\beta_2)(1+\beta_1)}{\beta_1-\beta_2}\left(\frac{1-2\gamma K_F-\gamma K_D}{c}\right)^{\beta_1-1}X^{\beta_1}<0.
$$

- Region 3: $X \ge c/(1-\gamma K_D-2\gamma K_F)$.

  In this region, we have

$$
\begin{aligned}
& V_F\left(X,K_D,K_F\right)-\delta K_F \\
=& N_2\left(K_D,K_F\right)X^{\beta_2}+\frac{K_F-\gamma K_D K_F-\gamma K_F^2}{r-\alpha}X-\frac{cK_F}{r}-\delta K_F \\
=& \frac{c^{1-\beta_2}\left[\left(1-\gamma K_D\right)^{1+\beta_2}-\left(1-2\gamma K_F-\gamma K_D\right)^{1+\beta_2}\right]F\left(\beta_1\right)}{4\gamma(\beta_1-\beta_2)}X^{\beta_2} \\
& +\frac{K_F-\gamma K_D K_F-\gamma K_F^2}{r-\alpha}X-\frac{cK_F}{r}-\delta K_F .
\end{aligned}
$$

Taking the first order condition with respect to K_F yields $K_F(X,K_D)$ must satisfy

$$\frac{F(\beta_1)\,c^{1-\beta_2}\,(1+\beta_2)\,(1-2\gamma K_F-\gamma K_D)^{\beta_2}}{2(\beta_1-\beta_2)}X^{\beta_2}+\frac{1-2\gamma K_F-\gamma K_D}{r-\alpha}X-\frac{c}{r}-\delta=0. \tag{3.38}$$

The second order partial derivative with respect to K_F is

$$\begin{aligned}
&-\frac{2\gamma\beta_2(1+\beta_2)c^{1-\beta_2}F(\beta_1)(1-2\gamma K_F-\gamma K_D)^{\beta_2-1}}{2(\beta_1-\beta_2)}X^{\beta_2}-\frac{2\gamma X}{r-\alpha}\\
\leq&-\frac{\gamma\beta_2(1+\beta_2)c^{1-\beta_2}F(\beta_1)(1-2\gamma K_F-\gamma K_D)^{\beta_2-1}}{\beta_1-\beta_2}\left(\frac{c}{1-\gamma K_D-2\gamma K_F}\right)^{\beta_2}\\
&-\frac{2\gamma}{r-\alpha}\frac{c}{1-\gamma K_D-2\gamma K_F}\\
=&-\frac{\gamma\beta_2(1+\beta_2)F(\beta_1)}{\beta_1-\beta_2}\frac{c}{1-\gamma K_D-2\gamma K_F}-\frac{2\gamma}{r-\alpha}\frac{c}{1-\gamma K_D-2\gamma K_F}\\
=&\frac{\gamma c}{1-\gamma K_D-2\gamma K_F}\left[-\frac{\beta_2(1+\beta_2)F(\beta_1)}{\beta_1-\beta_2}-\frac{2}{r-\alpha}\right]\\
<&0,
\end{aligned}$$

with the last step being concluded from the Appendix in Chapter 2.

The optimal investment threshold $X_F^*(K_D)$ in each region can be derived by the value matching and smooth pasting conditions at $X_F^*(K_D)$:

$$\begin{cases}
AX_F^{*\beta_1}(K_D) &= V_F\left(X_F^*(K_D),K_D,K_F\left(X_F^*(K_D),K_D\right)\right)-\delta K_F\left(X_F^*(K_D),K_D\right),\\
\beta_1 AX_F^{*\beta_1-1}(K_D) &= \frac{d}{dX}\left[V_F\left(X_F^*(K_D),K_D,K_F\left(X_F^*(K_D),K_D\right)\right)-\delta K_F\left(X_F^*(K_D),K_D\right)\right].
\end{cases}$$

Thus, $X_F^*(K_D)$ satisfies the implicit equation,

$$\begin{aligned}
&V_F\left(X_F,K_D,K_F\left(X_F,K_D\right)\right)-\delta K_F\left(X_F,K_D\right)\\
=&\frac{X_F(K_D)}{\beta_1}\frac{d\left[V_F\left(X_F,K_D,K_F\left(X_F,K_D\right)\right)-\delta K_F\left(X_F,K_D\right)\right]}{dX}.
\end{aligned}$$

- Region 1
  The optimal investment threshold $X_F^*(K_D)$ satisfies the following equation:

$$\frac{c^{1-\beta_1}\left[(1-\gamma K_D)^{1+\beta_1}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}\right]F(\beta_2)}{4\gamma(\beta_1-\beta_2)}X^{\beta_1}-\delta K_F$$

$$=\frac{X}{\beta_1}\beta_1 X^{\beta_1-1}\frac{c^{1-\beta_1}\left[(1-\gamma K_D)^{1+\beta_1}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}\right]F(\beta_2)}{4\gamma(\beta_1-\beta_2)},$$

which is equivalent to

$$\delta K_F = 0. \tag{3.39}$$

- Region 2

  The optimal threshold $X_F^*(K_D)$ satisfies

$$\begin{aligned}
&-\frac{F(\beta_2)c^{1-\beta_1}(1-2\gamma K_F-\gamma K_D)^{1+\beta_1}X^{\beta_1}}{4\gamma(\beta_1-\beta_2)}+\frac{c^{1-\beta_2}(1-\gamma K_D)^{1+\beta_2}}{4\gamma(\beta_1-\beta_2)}F(\beta_1)X^{\beta_2}\\
&+\frac{(1-\gamma K_D)^2X}{4\gamma(r-\alpha)}-\frac{c(1-\gamma K_D)}{2\gamma r}+\frac{c^2}{4\gamma X(r+\alpha-\sigma^2)}-\delta K_F\\
=&-\frac{F(\beta_2)c^{1-\beta_1}(1-\gamma K_D-2\gamma K_F)^{1+\beta_1}X^{\beta_1}}{4\gamma(\beta_1-\beta_2)}\\
&+\frac{X}{\beta_1}\left[\frac{\beta_2F(\beta_1)c^{1-\beta_2}(1-\gamma K_D)^{1+\beta_2}X^{\beta_2-1}}{4\gamma(\beta_1-\beta_2)}+\frac{(1-\gamma K_D)^2}{4\gamma(r-\alpha)}-\frac{c^2}{4\gamma X^2(r+\alpha-\sigma^2)}\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{c^{1-\beta_2}(1-\gamma K_D)^{1+\beta_2}F(\beta_1)X^{\beta_2}}{4\gamma\beta_1}+\frac{1}{4\gamma}\left[\frac{\beta_1-1}{\beta_1}\frac{(1-\gamma K_D)^2X}{r-\alpha}-\frac{2c(1-\gamma K_D)}{r}\right.\\
&\left.+\frac{\beta_1+1}{\beta_1}\frac{c^2}{X(r+\alpha-\sigma^2)}\right]-\delta K_F=0.
\end{aligned} \tag{3.40}$$

- Region 3

  The optimal investment threshold $X_F^*(K_D)$ satisfies

$$\begin{aligned}
&\frac{c^{1-\beta_2}\left[(1-\gamma K_D)^{1+\beta_2}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_2}\right]F(\beta_1)}{4\gamma(\beta_1-\beta_2)}X^{\beta_2}\\
&+\frac{K_F-\gamma K_DK_F-\gamma K_F^2}{r-\alpha}X-\frac{cK_F}{r}-\delta K_F\\
=&\frac{\beta_2X^{\beta_2}}{\beta_1}\frac{c^{1-\beta_2}\left[(1-\gamma K_D)^{1+\beta_2}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_2}\right]F(\beta_1)}{4\gamma(\beta_1-\beta_2)}\\
&+\frac{X}{\beta_1}\frac{K_F-\gamma K_DK_F-\gamma K_F^2}{r-\alpha}.
\end{aligned}$$

Rearranging terms yields

$$\frac{X^{\beta_2}c^{1-\beta_2}F(\beta_1)}{4\gamma\beta_1}\left[(1-\gamma K_D)^{1+\beta_2}-(1-2\gamma K_F-\gamma K_D)^{1+\beta_2}\right]$$
$$+\frac{(\beta_1-1)X}{\beta_1}\frac{K_F-\gamma K_D K_F-\gamma K_F^2}{r-\alpha}-\frac{cK_F}{r}-\delta K_F=0. \tag{3.41}$$

Note that in the monopoly case, whether the flexible firm produces up to capacity depends on the economic setting. Next, we are going to examine the conditions for the flexible follower to produce below and up to capacity.

If the firm produces below capacity right after investment, then $K_F(X,K_D)>q_F(X,K_D,K_F(X,K_D))$, i.e.,

$$\frac{1}{2\gamma}\left\{1-\gamma K_D-\frac{c}{X}\left[\frac{2\delta(\beta_1-\beta_2)}{cF(\beta_2)(1+\beta_1)}\right]^{\frac{1}{\beta_1}}\right\}>\frac{X(1-\gamma K_D)-c}{2\gamma X}.$$

It is equivalent to

$$2\delta(\beta_1-\beta_2)<cF(\beta_2)(1+\beta_1), \tag{3.42}$$

which is the same as in the monopoly case. Furthermore, we can deduce that

$$2\delta(\beta_1-\beta_2)\geq cF(\beta_2)(1+\beta_1) \tag{3.43}$$

would define Region 3, where the firm produces up to capacity right after investment. The definitions of Region 2, equation (3.42), and Region 3, equation (3.43), for the flexible follower firm are the same as that for the monopoly flexible firm in Chapter 2.

Proof of Corollary 3.1

- Region 2
  Derive $dX_F^*(K_D)/dK_D$ and check whether the leader's installed capacity level would delay the flexible follower's investment. Dividing (3.10) by $1-\gamma K_D$, we get

$$\frac{c^{1-\beta_2}F(\beta_1)}{4\gamma\beta_1}[X(1-\gamma K_D)]^{\beta_2}-\frac{\delta}{2\gamma}\left\{1-\frac{c}{X(1-\gamma K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}$$
$$+\frac{1}{4\gamma}\left[\frac{\beta_1-1}{\beta_1}\frac{X(1-\gamma K_D)}{r-\alpha}-\frac{2c}{r}+\frac{\beta_1+1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{X(1-\gamma K_D)}\right]=0. \tag{3.44}$$

Comparing (3.44) with the implicit equation that determines the optimal investment threshold in the corresponding monopoly model, see Chapter 2, we find that $X(1-\gamma K_D)$ replaces X^* in the corresponding monopoly case. Denote $x(K_D) = X_F^*(K_D)(1-\gamma K_D)$, which is not smaller than c according to the follower's value function, and (3.44) can be rewritten as

$$\frac{c^{1-\beta_2}F(\beta_1)}{\beta_1}[x(K_D)]^{\beta_2} - 2\delta\left\{1 - \frac{c}{x(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}$$
$$+\frac{\beta_1-1}{\beta_1}\frac{x(K_D)}{r-\alpha} - \frac{2c}{r} + \frac{\beta_1+1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{x(K_D)} = 0.$$

Taking the derivative with respect to K_D yields

$$\left\{\frac{c^{1-\beta_2}\beta_2 F(\beta_1)}{\beta_1}[x(K_D)]^{\beta_2-1} - \frac{2c\delta}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right.$$
$$\left.+\frac{\beta_1-1}{\beta_1(r-\alpha)} - \frac{\beta_1+1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{x^2(K_D)}\right\}\frac{dx(K_D)}{dK_D} = 0. \tag{3.45}$$

First check that whether the coefficient of $dX(K_D)/dK_D$ equals to 0. This coefficient can be rewritten as

$$\begin{aligned}
Y(x(K_D)) =& \frac{\beta_2}{x(K_D)}\left\{2\delta - \frac{2c\delta}{x(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}} - \frac{\beta_1-1}{\beta_1}\frac{x(K_D)}{r-\alpha}\right.\\
&\left.+\frac{2c}{r} - \frac{\beta_1+1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{x(K_D)}\right\} - \frac{2c\delta}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\\
&+\frac{\beta_1-1}{\beta_1(r-\alpha)} - \frac{\beta_1+1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{x^2(K_D)}\\
=&\frac{2\delta\beta_2}{x(K_D)} - \frac{2c\delta\beta_2}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}} - \frac{2c\delta}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\\
&-\frac{(\beta_1-1)(\beta_2-1)}{\beta_1(r-\alpha)} + \frac{2c\beta_2}{rx(K_D)} - \frac{(\beta_1+1)(\beta_2+1)}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{1}{x^2(K_D)}\\
=&\frac{2\delta\beta_2}{x(K_D)} - \frac{2c\delta(\beta_2+1)}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}} - \frac{1}{\beta_1(r-\alpha)}\frac{2(\alpha-r)}{\sigma^2}\\
&+\frac{2c\beta_2}{rx(K_D)} - \frac{1}{\beta_1}\frac{c^2}{r+\alpha-\sigma^2}\frac{2(\sigma^2-r-\alpha)}{\sigma^2}\frac{1}{x^2(K_D)}\\
=&\frac{2\delta\beta_2}{x(K_D)} - \frac{2c\delta(\beta_2+1)}{x^2(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}} + \frac{2}{\beta_1\sigma^2} + \frac{2c\beta_2}{rx(K_D)}
\end{aligned}$$

$$+\frac{2}{\beta_1\sigma^2}\frac{c^2}{x^2(K_D)}, \tag{3.46}$$

with $x(K_D) \geq c$. (3.46) is a quadratic form of $1/x(K_D)$. The discriminant is

$$
\begin{aligned}
&\left(\frac{2c\beta_2}{r}+2\delta\beta_2\right)^2-\frac{8}{\beta_1\sigma^2}\left\{\frac{2c^2}{\beta_1\sigma^2}-2c\delta(1+\beta_2)\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}\\
=&4\left\{\beta_2^2\left(\delta+\frac{c}{r}\right)^2-\frac{4c^2}{\beta_1^2\sigma^4}+\frac{4c\delta(1+\beta_2)}{\beta_1\sigma^2}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}\\
=&\frac{4}{\beta_1^2\sigma^4}\left\{\beta_1^2\beta_2^2\sigma^4\left(\delta+\frac{c}{r}\right)^2-4c^2+4c\delta\sigma^2(\beta_1+\beta_1\beta_2)\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}\\
=&\tfrac{4}{\beta_1^2\sigma^4}\left\{\tfrac{4r^2\sigma^4}{\sigma^4}\left(\delta^2+\tfrac{2c\delta}{r}+\tfrac{c^2}{r^2}\right)-4c^2+\left(4c\delta\sigma^2\beta_1-4c\delta\sigma^2\tfrac{2r}{\sigma^2}\right)\left[\tfrac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}\\
=&\frac{4}{\beta_1^2\sigma^4}\left\{4c\delta\sigma^2\beta_1\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}+8rc\delta-8rc\delta\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}+4r^2\delta^2\right\}\\
>&0.
\end{aligned}
$$

Besides,

$$
\begin{aligned}
Y(c)=&\frac{2\delta\beta_2}{c}-\frac{2\delta\beta_2}{c}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}-\frac{2\delta}{c}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}+\frac{4}{\beta_1\sigma^2}+\frac{2\beta_2}{r}\\
=&\frac{2\delta\beta_2}{c}\left\{1-\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}-\frac{2\delta}{c}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}+\frac{4r+2\sigma^2\beta_1\beta_2}{r\beta_1\sigma^2}\\
=&\frac{2\delta\beta_2}{c}\left\{1-\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}-\frac{2\delta}{c}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}+\frac{4r-4r}{r\beta_1\sigma^2}\\
<&0.
\end{aligned}
$$

There are at most two positive values of x bigger than c that make $Y(x(K_D))=0$. If $Y(x(K_D))\neq 0$ and for (3.45) to hold, it must be $dx(K_D)/dK_D=0$. If $Y(x(K_D))=0$, then $x(K_D)$ is a constant, which also means that $dx(K_D)/dK_D=0$. Thus, it can be concluded that

$$\frac{dx(K_D)}{dK_D}=\frac{dX_F^*(K_D)}{dK_D}(1-\gamma K_D)-\gamma X_F^*(K_D)=0,$$

which leads to

$$\frac{dX_F^*(K_D)}{dK_D} = \frac{\gamma X_F^*(K_D)}{1-\gamma K_D} > 0, \tag{3.47}$$

implying that investing in more capacity by the dedicated leader would delay the investment of the flexible follower. According to (3.9), taking the derivative of $K_F^*(K_D)$ with respect to K_D, we get

$$\begin{aligned}
\frac{dK_F^*(K_D)}{dK_D} =& \frac{1}{2\gamma}\left\{-\gamma + \frac{c}{X_F^{*2}(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\frac{dX_F^*(K_D)}{dK_D}\right\}\\
=& \frac{1}{2}\left\{-1 + \frac{c}{(1-\gamma K_D)X_F^*(K_D)}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right\}\\
=& -\frac{\gamma K_F^*(K_D)}{1-\gamma K_D} \leq 0.
\end{aligned} \tag{3.48}$$

This implies that an increase in the inflexible leader's investment capacity decreases the flexible follower's optimal capacity to invest with.

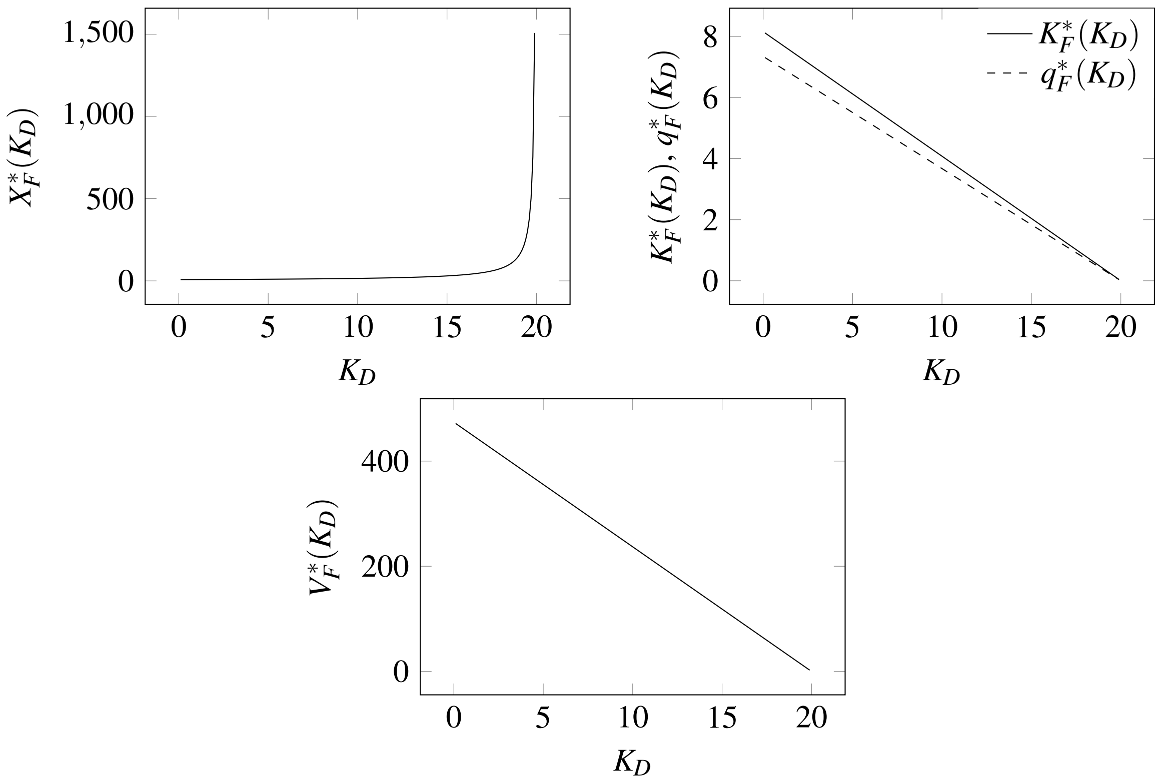


Figure 3.18: Illustration of $X_F^*(K_D)$, $K_F^*(K_D)$, $q_F^*(K_D)$, and $V_F^*(K_D)$ when the follower produces below capacity right after the investment. Parameter values are $\alpha = 0.04$, $r = 0.1$, $\sigma = 0.3$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

Figure 3.18 illustrates how the flexible follower's optimal investment threshold $X_F^*(K_D)$, the optimal investment capacity $K_F^*(K_D)$, the output level right after the investment $q_F^*(K_D) = q_F(X_F^*(K_D), K_D, K_F^*(K_D))$, and the project value when the firm invests $V_F^*(K_D) = V_F(X_F^*(K_D), K_D, K_F^*(K_D))$, change with the inflexible leader's investment capacity K_D when the firm produces below capacity right after the investment. It illustrates that the increase in the leader's investment capacity indeed delays the flexible follower's investment and decreases the capacity with witch the follower enters the market. In fact, if the leader invests with a capacity that is very close to the market size, then the flexible follower can be almost kept out of the market.

- Region 3
  Check whether the leader's capacity influences the investment decision of the flexible follower. The investment timing $X_F^*(K_D)$ and investment capacity $K_F^*(K_D)$ are determined by (3.11) and (3.12) when the follower produces up to capacity right after the investment. Rewriting these two equations yields

$$\frac{F(\beta_1)c^{1-\beta_2}(1+\beta_2)}{2(\beta_1-\beta_2)}[z(K_D)]^{\beta_2} + \frac{1}{r-\alpha}z(K_D) - \frac{c}{r} - \delta = 0, \tag{3.49}$$

and

$$\frac{c^{1-\beta_2}F(\beta_1)}{4\gamma\beta_1 X_F^*(K_D)}\left\{[w(K_D)]^{1+\beta_2} - [z(K_D)]^{1+\beta_2}\right\} - \frac{cK_F^*(K_D)}{r} - \delta K_F^*(K_D)$$
$$+ \frac{\beta_1 - 1}{4\gamma\beta_1(r-\alpha)X_F^*(K_D)}\left[w^2(K_D) - z^2(K_D)\right] = 0, \tag{3.50}$$

respectively, where

$$w(K_D) = X_F^*(K_D)(1-\gamma K_D),$$
$$z(K_D) = X_F^*(K_D)(1-\gamma K_D - 2\gamma K_F^*(K_D)).$$

Taking the derivative of (3.49) with respect to K_D yields for all $K_D \geq 0$

$$\left[\frac{F(\beta_1)c^{1-\beta_2}\beta_2(1+\beta_2)[z(K_D)]^{\beta_2-1}}{2(\beta_1-\beta_2)} + \frac{1}{r-\alpha}\right]\frac{dz(K_D)}{dK_D} = 0. \tag{3.51}$$

For (3.51) to hold, it should be that $dz(K_D)/dK_D = 0$, implying $z(K_D)$ is a constant. If $dz(K_D)/dK_D \neq 0$, then $z(K_D)$ changes with K_D, and (3.51) cannot hold for all K_D, a contradiction. From $dz(K_D)/dK_D = 0$, it follows that

$$\frac{dX_F^*(K_D)}{dK_D} = \frac{\gamma X_F^*(K_D)}{1-\gamma K_D - 2\gamma K_F^*(K_D)}\left(1+2\frac{dK_F^*(K_D)}{dK_D}\right). \tag{3.52}$$

Taking the derivative of (3.50) with respect to K_D gives

$$\begin{aligned}
&-\frac{c^{1-\beta_2}F(\beta_1)}{4\gamma\beta_1 X_F^{*2}(K_D)}\left\{[w(K_D)]^{1+\beta_2}-[z(K_D)]^{1+\beta_2}\right\}\frac{dX_F^*(K_D)}{dK_D}\\
&+\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)}{4\gamma\beta_1 X_F^*(K_D)}[w(K_D)]^{\beta_2}\frac{dw(K_D)}{dK_D}\\
&-\frac{\beta_1-1}{4\gamma(r-\alpha)\beta_1 X_F^{*2}(K_D)}\left[w^2(K_D)-z^2(K_D)\right]\frac{dX_F^*(K_D)}{dK_D}\\
&+\frac{(\beta_1-1)w(K_D)}{2\gamma(r-\alpha)\beta_1 X_F^*(K_D)}\frac{dw(K_D)}{dK_D}-\left(\frac{c}{r}+\delta\right)\frac{dK_F^*(K_D)}{dK_D}\\
=&-\left(\frac{c}{r}+\delta\right)\left(\frac{K_F^*(K_D)}{X_F^*(K_F}\frac{dX_F^*(K_D)}{dK_D}+\frac{dK_F^*(K_D)}{dK_D}\right)\\
&+\left\{\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2}}{4\gamma\beta_1 X_F^*(K_D)}+\frac{(\beta_1-1)w(K_D)}{2\gamma(r-\alpha)\beta_1 X_F^*(K_D)}\right\}\frac{dw(K_D)}{dK_D}\\
=&-\left(\frac{c}{r}+\delta\right)\left[\frac{\gamma K_F^*(K_D)}{1-\gamma K_D-2\gamma K_F^*(K_D)}\left(1+2\frac{dK_F^*(K_D)}{dK_D}\right)+\frac{dK_F^*(K_D)}{dK_D}\right]\\
&+\left[(1-\gamma K_D)\frac{dX_F^*(K_D)}{dK_D}-\gamma X_F^*(K_D)\right]\left\{\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2}}{4\gamma\beta_1 X_F^*(K_D)}\right.\\
&\left.+\frac{(\beta_1-1)w(K_D)}{2\gamma(r-\alpha)\beta_1 X_F^*(K_D)}\right\}\\
=&-\left(\frac{c}{r}+\delta\right)\frac{\gamma K_F^*(K_D)+(1-\gamma K_D)\frac{dK_F^*(K_D)}{dK_D}}{1-\gamma K_D-2\gamma K_F^*(K_D)}+\gamma X_F^*(K_D)\left[\frac{1-\gamma K_D}{1-\gamma K_D-2\gamma K_F^*(K_D)}\left(1+\right.\right.\\
&\left.\left.2\frac{dK_F^*(K_D)}{dK_D}\right)-1\right]\left\{\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2}}{4\gamma\beta_1 X_F^*(K_D)}+\frac{(\beta_1-1)w(K_D)}{2\gamma(r-\alpha)\beta_1 X_F^*(K_D)}\right\}\\
=&-\left(\frac{c}{r}+\delta\right)\frac{\gamma K_F^*(K_D)+(1-\gamma K_D)\frac{dK_F^*(K_D)}{dK_D}}{1-\gamma K_D-2\gamma K_F^*(K_D)}\\
&+\frac{\gamma K_F^*(K_D)+(1-\gamma K_D)\frac{dK_F^*(K_D)}{dK_D}}{1-\gamma K_D-2\gamma K_F^*(K_D)}\left\{\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2}}{2\beta_1}+\frac{(\beta_1-1)w(K_D)}{\beta_1(r-\alpha)}\right\}\\
=&0.
\end{aligned}$$

Thus,

$$\frac{c}{r}+\delta=\frac{(1+\beta_2)c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2}}{2\beta_1}+\frac{(\beta_1-1)w(K_D)}{\beta_1(r-\alpha)}. \tag{3.53}$$

Taking the derivative of (3.53) with respect to K_D yields

$$\left\{\frac{(1+\beta_2)\beta_2c^{1-\beta_2}F(\beta_1)[w(K_D)]^{\beta_2-1}}{2\beta_1}+\frac{\beta_1-1}{\beta_1(r-\alpha)}\right\}\frac{dw(K_D)}{dK_D}=0. \tag{3.54}$$

Similar to that (3.51) implies $z(K_D)$ is a constant, the fact that (3.54) holds implies $w(K_D)$ is also a constant and satisfies

$$\frac{dw(K_D)}{dK_D}=-\gamma X_F^*(K_D)+(1-\gamma K_D)\frac{dX_F^*(K_D)}{dK_D}=0.$$

It can be further derived that

$$\frac{dX_F^*(K_D)}{dK_D}=\frac{\gamma X_F^*(K_D)}{1-\gamma K_D}>0. \tag{3.55}$$

Moreover, from (3.52) and (3.55), we get

$$\frac{dK_F^*(K_D)}{dK_D}=-\frac{\gamma K_F^*(K_D)}{1-\gamma K_D}<0. \tag{3.56}$$

Thus, for the case that the flexible follower produces up to capacity right after the investment, the inflexible leader can delay and decrease the investment of the follower by investing in a larger capacity.

Figure 3.19 illustrates the flexible follower's optimal investment threshold $X_F^*(K_D)$, the optimal investment capacity $K_F^*(K_D)$, and the project value at the moment of investment $V_F^*(K_D)$ as functions of K_D. It confirms that $X_F^*(K_D)$ increases with K_D, while $K_F^*(K_D)$ and $V_F^*(K_D)$ decrease with K_D. Thus, the leader can delay the investment of the flexible follower. The leader can even prevent the follower entering the market by investing with a capacity close to the market boundary.

Derivation of $\mathcal{L}(K_D), \mathcal{M}_1(K_D), \mathcal{M}_2(K_D), \mathcal{N}(K_D)$ Employing value matching and smooth

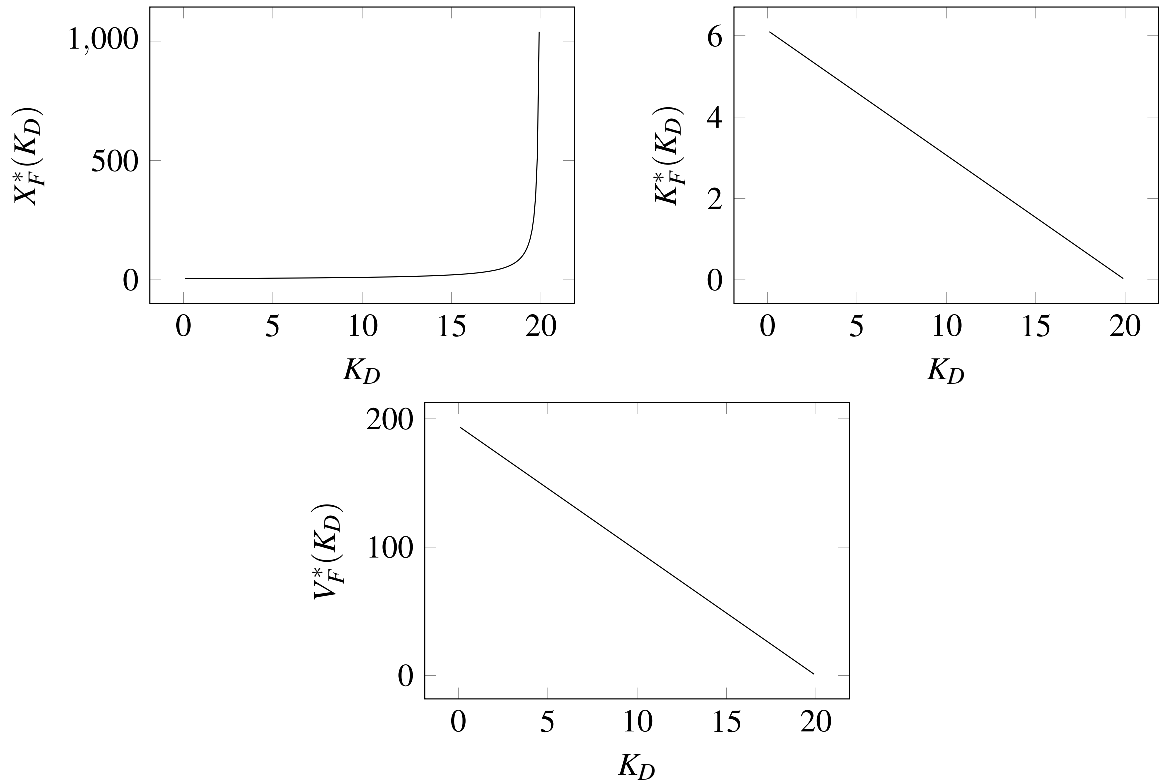


Figure 3.19: Illustration of $X_F^*(K_D)$, $K_F^*(K_D)$, and $V_F^*(K_D)$ when the follower produces up to capacity right after investment. Parameter values are $\alpha = 0.03$, $r = 0.1$, $\sigma = 0.14$, $\gamma = 0.05$, $c = 2$, $\delta = 10$.

pasting at $X_1 = \frac{c}{1-\gamma K_D}$ and $X_2 = \frac{c}{1-\gamma K_D - 2\gamma K_F^*(K_D)}$, then given K_D $(0 \le K_D < 1/\gamma)$,

$$\mathcal{L}(K_D)X_1^{\beta_1} + \frac{K_D(1-\gamma K_D)}{r-\alpha}X_1 - \frac{cK_D}{r} = \mathcal{M}_1(K_D)X_1^{\beta_1} + \mathcal{M}_2(K_D)X_1^{\beta_2} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X_1 - \frac{cK_D}{2r} \tag{3.57}$$

and

$$\beta_1\mathcal{L}(K_D)X_1^{\beta_1-1} + \frac{K_D(1-\gamma K_D)}{r-\alpha} = \beta_1\mathcal{M}_1(K_D)X_1^{\beta_1-1} + \beta_2\mathcal{M}_2(K_D)X_1^{\beta_2-1} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}, \tag{3.58}$$

$$\begin{aligned}\mathcal{M}_1(K_D)X_2^{\beta_1} + \mathcal{M}_2(K_D)X_2^{\beta_2} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X_2 - \frac{cK_D}{2r} \\ = \mathcal{N}(K_D)X_2^{\beta_2} + \frac{K_D(1-\gamma K_D - \gamma K_F^*(K_D))}{r-\alpha}X_2 - \frac{cK_D}{r}\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\beta_1\mathcal{M}_1(K_D)X_2^{\beta_1-1} + \beta_2\mathcal{M}_2(K_D)X_2^{\beta_2-1} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)} \\ = \beta_2\mathcal{N}(K_D)X_2^{\beta_2-1} + \frac{K_D(1-\gamma K_D - \gamma K_F^*(K_D))}{r-\alpha}.\end{aligned} \tag{3.60}$$

From (3.57), it holds that

$$\mathcal{L}(K_D)X_1^{\beta_1} - \mathcal{M}_1(K_D)X_1^{\beta_1} = \mathcal{M}_2(K_D)X_1^{\beta_2} - \frac{K_D(1-\gamma K_D)X_1}{2(r-\alpha)} + \frac{cK_D}{2r}. \tag{3.61}$$

From (3.58), it stands that

$$\beta_1\left(\mathcal{L}(K_D)X_1^{\beta_1} - \mathcal{M}_1(K_D)X_1^{\beta_1}\right) = \beta_2\mathcal{M}_2(K_D)X_1^{\beta_2} - \frac{K_D(1-\gamma K_D)X_1}{2(r-\alpha)}.$$

Thus, it can be derived that

$$\begin{aligned}\mathcal{M}_2(K_D) &= \frac{X_1^{-\beta_2}}{\beta_1-\beta_2}\left[\frac{K_D(\beta_1-1)(1-\gamma K_D)X_1}{2(r-\alpha)} - \frac{\beta_1 cK_D}{2r}\right] \\ &= \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2}.\end{aligned} \tag{3.62}$$

From (3.59), it holds that

$$\mathcal{N}(K_D)X_2^{\beta_2} - \mathcal{M}_2(K_D)X_2^{\beta_2} = \mathcal{M}_1(K_D)X_2^{\beta_1} - \frac{K_D(1-\gamma K_D - 2\gamma K_F^*(K_D))X_2}{2(r-\alpha)} + \frac{cK_D}{2r}. \quad (3.63)$$

From (3.60), it stands that

$$\beta_2\mathcal{N}(K_D)X_2^{\beta_2} - \beta_2\mathcal{M}_2(K_D)X_2^{\beta_2} = \beta_1\mathcal{M}_1(K_D)X_2^{\beta_1} - \frac{K_D(1-\gamma K_D - 2\gamma K_F^*(K_D))X_2}{2(r-\alpha)}.$$

Thus, it can be derived that

$$\begin{aligned}
\mathcal{M}_1(K_D) &= \frac{X_2^{-\beta_1}}{\beta_1-\beta_2}\left[\frac{(1-\beta_2)K_D(1-\gamma K_D-2\gamma K_F^*(K_D))X_2}{2(r-\alpha)} + \frac{\beta_2 cK_D}{2r}\right] \\
&= \frac{X_2^{-\beta_1}}{\beta_1-\beta_2}\left[\frac{(1-\beta_2)cK_D}{2(r-\alpha)} + \frac{\beta_2 cK_D}{2r}\right] \\
&= \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{1-\beta_2}{r-\alpha} + \frac{\beta_2}{r}\right)\left(\frac{c}{1-\gamma K_D-2\gamma K_F^*(K_D)}\right)^{-\beta_1} \\
&= -\frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha} - \frac{\beta_2}{r}\right)\left(\frac{c}{1-\gamma K_D-2\gamma K_F^*(K_D)}\right)^{-\beta_1}. \qquad (3.64)
\end{aligned}$$

From (3.61), it can be derived that

$$\begin{aligned}
\mathcal{L}(K_D) &= \mathcal{M}_1(K_D) + \mathcal{M}_2(K_D)X_1^{\beta_2-\beta_1} - \frac{cK_D X_1^{-\beta_1}}{2}\left(\frac{1}{r-\alpha} - \frac{1}{r}\right) \\
&= \mathcal{M}_1(K_D) + \frac{cK_D X_1^{-\beta_1}}{2}\left[\frac{\beta_1-1}{(r-\alpha)(\beta_1-\beta_2)} - \frac{\beta_1}{r(\beta_1-\beta_2)} - \frac{1}{r-\alpha} + \frac{1}{r}\right] \\
&= \mathcal{M}_1(K_D) + \frac{cK_D X_1^{-\beta_1}}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha} - \frac{\beta_2}{r}\right) \\
&= \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha} - \frac{\beta_2}{r}\right)\left(X_1^{-\beta_1} - X_2^{-\beta_1}\right) \\
&= \frac{c^{1-\beta_1}K_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha} - \frac{\beta_2}{r}\right)\left[(1-\gamma K_D)^{\beta_1} - (1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_1}\right]. \quad (3.65)
\end{aligned}$$

From (3.63), it can be derived that

$$\mathcal{N}(K_D) = \mathcal{M}_1(K_D)X_2^{\beta_1-\beta_2} + \mathcal{M}_2 + X_2^{-\beta_2}\left(\frac{cK_D}{2r} - \frac{cK_D}{2(r-\alpha)}\right)$$

$$
\begin{aligned}
&= \mathcal{M}_2(K_D) + \frac{cK_D X_2^{-\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{1-\beta_2}{r-\alpha}+\frac{\beta_2}{r}\right) + \frac{cK_D X_2^{-\beta_2}}{2}\left(\frac{1}{r}-\frac{1}{r-\alpha}\right)\\
&= \mathcal{M}_2(K_D) + \frac{cK_D X_2^{-\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{1-\beta_1}{r-\alpha}+\frac{\beta_1}{r}\right)\\
&= \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(X_1^{-\beta_2}-X_2^{-\beta_2}\right)\\
&= \frac{c^{1-\beta_2}K_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[(1-\gamma K_D)^{\beta_2}-(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}\right]. \quad (3.66)
\end{aligned}
$$

Next, we look at the signs for $\mathcal{L}(K_D)$, $\mathcal{M}_1(K_D)$, $\mathcal{M}_2(K_D)$, and $\mathcal{N}(K_D)$. In order to do this, we first check the signs of $(\beta-1)/(r-\alpha)-\beta/r = \frac{\alpha\beta-r}{r(r-\alpha)}$ for $\beta=\beta_1$ and $\beta=\beta_2$. If $\alpha \geq 0$, then $\alpha\beta_2 - r < 0$ because $\beta_2 < 0$. If $\alpha < 0$, then

$$
\alpha\beta_2 - r = \alpha\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}-\frac{r}{\alpha}-\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\right),
$$

with $\frac{1}{2}-\frac{\alpha}{\sigma^2}-\frac{r}{\alpha} > 0$. From

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}-\frac{r}{\alpha}\right)^2-\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2-\frac{2r}{\sigma^2}\\
&= -\frac{2r}{\alpha}\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)+\frac{r^2}{\alpha^2}-\frac{2r}{\sigma^2}\\
&= -\frac{r}{\alpha}+\frac{r^2}{\alpha^2} > 0,
\end{aligned}
$$

we get $\alpha\beta_2 - r < 0$. So, $\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r} < 0$.

If $\alpha \leq 0$, then $\alpha\beta_1 - r < 0$. If $\alpha > 0$, then

$$
\alpha\beta_1 - r = \alpha\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}-\frac{r}{\alpha}+\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}\right),
$$

with $\frac{1}{2}-\frac{\alpha}{\sigma^2}-\frac{r}{\alpha} < 0$, because $r > \alpha$. From

$$
\left(\frac{r}{\alpha}+\frac{\alpha}{\sigma^2}-\frac{1}{2}\right)^2-\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2-\frac{2r}{\sigma^2}
$$

$$
\begin{aligned}
&= \frac{r^2}{\alpha^2} + \frac{2r}{\alpha}\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right) - \frac{2r}{\sigma^2} \\
&= \frac{r^2}{\alpha^2} - \frac{r}{\alpha} > 0,
\end{aligned}
$$

we get $\alpha\beta_1 - r < 0$. So, $\frac{\beta_1 - 1}{r-\alpha} - \frac{\beta_1}{r} < 0$.

Thus, it can be concluded that for $0 \le K_D < 1/\gamma$

$$
\begin{aligned}
\mathcal{L}(K_D) &< 0, \\
\mathcal{M}_1(K_D) &> 0, \\
\mathcal{M}_2(K_D) &< 0, \\
\mathcal{N}(K_D) &> 0.
\end{aligned}
$$

Proof of Negative $\mathcal{B}_1(K_D)$ Before the derivation of the dedicated leader's optimal investment capacity in the entry deterrence and accommodation strategies, we first look at the sign of $\mathcal{B}_1(K_D)$. We have

$$
\begin{aligned}
\mathcal{B}_1(K_D) =& \mathcal{M}_1(K_D) + \mathcal{M}_2(K_D) X_F^{*\beta_2 - \beta_1}(K_D) \\
& - \frac{K_D(1-\gamma K_D)}{2(r-\alpha)} X_F^{*1-\beta_1}(K_D) + \frac{cK_D}{2r} X_F^{*-\beta_1}(K_D) \\
=& \frac{1}{X_F^{*\beta_1}(K_D)} \left[-\frac{cK_D}{2(\beta_1-\beta_2)} \left(\frac{\beta_2 - 1}{r-\alpha} - \frac{\beta_2}{r}\right) \left(\frac{X_F^*(K_D)}{X_2(K_D)}\right)^{\beta_1} \right. \\
& \left. + \frac{cK_D}{2(\beta_1-\beta_2)} \left(\frac{\beta_1 - 1}{r-\alpha} - \frac{\beta_1}{r}\right) \left(\frac{X_F^*(K_D)}{X_1(K_D)}\right)^{\beta_2} - \frac{cK_D}{2(r-\alpha)} \frac{X_F^*(K_D)}{X_1(K_D)} + \frac{cK_D}{2r} \right] \\
=& \frac{cK_D}{2X_F^{*\beta_1}(K_D)} \left[-\frac{1}{\beta_1-\beta_2} \left(\frac{\beta_2 - 1}{r-\alpha} - \frac{\beta_2}{r}\right) \left(\frac{X_F^*(K_D)}{X_2(K_D)}\right)^{\beta_1} \right. \\
& \left. + \frac{1}{\beta_1-\beta_2} \left(\frac{\beta_1 - 1}{r-\alpha} - \frac{\beta_1}{r}\right) \left(\frac{X_F^*(K_D)}{X_1(K_D)}\right)^{\beta_2} - \frac{1}{r-\alpha} \frac{X_F^*(K_D)}{X_1(K_D)} + \frac{1}{r} \right].
\end{aligned}
$$

For $X_F^*(K_D)/X_1(K_D)$ and $X_F^*(K_D)/X_2(K_D)$, it holds that

$$
\frac{d}{dK_D} \frac{X_F^*(K_D)}{X_1(K_D)} = \frac{1}{X_1^2(K_D)} \left(\frac{\gamma K_F^*(K_D) X_1(K_D)}{1-\gamma K_D} - X_F^*(K_D) \frac{\gamma X_1(K_D)}{1-\gamma K_D} \right) = 0
$$

and

$$\frac{d}{dK_D}\frac{X_F^*(K_D)}{X_2(K_D)}=\frac{1}{X_2^2(K_D)}\left(\frac{\gamma K_F^*(K_D)X_2(K_D)}{1-\gamma K_D}-X_F^*(K_D)\frac{\gamma X_2(K_D)}{1-\gamma K_D}\right)=0.$$

This implies that $X_F^*(K_D)/X_1(K_D)$ and $X_F^*(K_D)/X_2(K_D)$ are constants and do not change with K_D. So we can set $K_D=0$, and then

$$\frac{X_F^*(K_D)}{X_1(K_D)}=\frac{X_F^*(0)}{c}$$

and

$$\frac{X_F^*(K_D)}{X_2(K_D)}=\frac{X_F^*(0)}{c}(1-2\gamma K_F^*(0))=\left(\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right)^{\frac{1}{\beta_1}}.$$

Equation (3.10) just becomes the corresponding implicit equation to determine X^* for the monopoly case:

$$F(\beta_1)\left(\frac{X^*}{c}\right)^{\beta_2}+\frac{\beta_1-1}{r-\alpha}\frac{X^*}{c}-\frac{2\beta_1}{r}+\frac{\beta_1+1}{r+\alpha-\sigma^2}\frac{c}{X^*}-\frac{2\beta_1\delta}{c}\left(1-\frac{c}{X^*}\left[\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right]^{\frac{1}{\beta_1}}\right)=0. \tag{3.67}$$

Recall

$$\begin{aligned}\mathcal{B}_1(K_D)=&\frac{cK_D}{2X_F^{*\beta_1}(K_D)}\left[-\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{X_F^*(K_D)}{X_2(K_D)}\right)^{\beta_1}\right.\\&\left.+\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X_F^*(K_D)}{X_1(K_D)}\right)^{\beta_2}-\frac{1}{r-\alpha}\frac{X_F^*(K_D)}{X_1(K_D)}+\frac{1}{r}\right]\\=&\frac{cK_D}{2(\beta_1-\beta_2)X_F^{*\beta_1}(K_D)}\left[-\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right.\\&\left.+\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X^*}{c}\right)^{\beta_2}-(\beta_1-\beta_2)\left(\frac{1}{r-\alpha}\frac{X^*}{c}-\frac{1}{r}\right)\right]\\=&\frac{cK_D}{2X_F^{*\beta_1}(K_D)}\mathcal{F}(X^*),\end{aligned}$$

where X^* satisfies (3.67). Next, we show $\mathcal{F}(X^*)$ is negative numerically. The demonstration is shown in Figure 3.20. Note that γ does not influence $\mathcal{F}(X^*)$, so the numerical analysis is just about the influence of α, σ, r, c, and δ. The default parameter values are

$\alpha = 0.05$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$. Some combination of parameter values does not make the flexible follower produce below capacity right after investment. After ruling out these combinations, $\mathcal{F}(X^*)$ changing with parameters is illustrated in Figure 3.20. The numerical analysis confirms that $\mathcal{B}_1(K_D)$ is negative when the flexible follower produces below capacity right after investment. In the following analysis, we will take $\mathcal{B}_1(K_D)$ as negative.

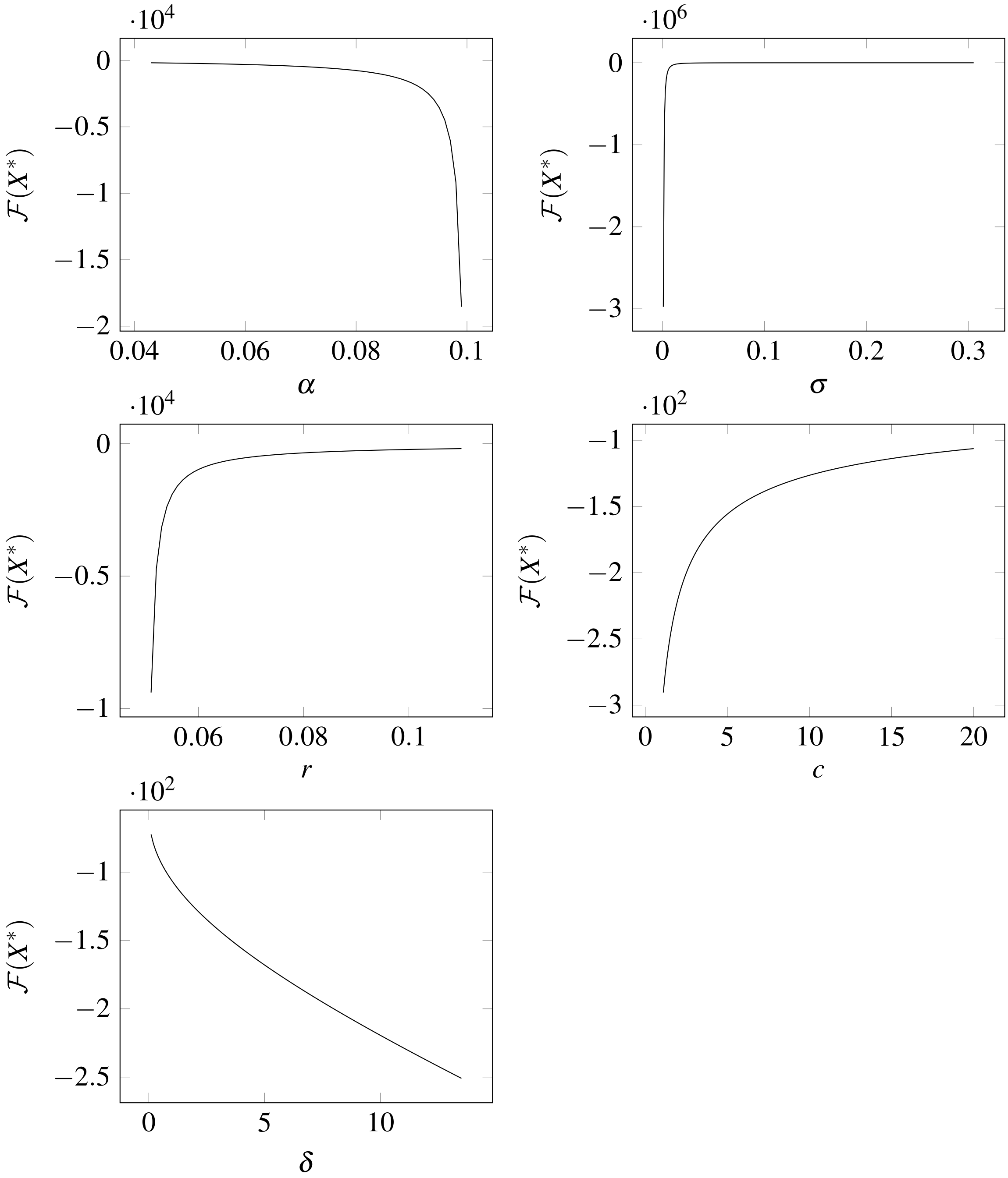


Figure 3.20: Illustration of negative $\mathcal{F}(X^*)$ changing with α, σ, r, c, and δ. Default parameter values are $\alpha = 0.05$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$.

Proof of Proposition 3.2 In order to get the optimal investment decisions for the dedicated leader, we first calculate the first derivative of $\mathcal{B}_1(K_D)$ with respect of K_D. First, $\mathcal{M}_1(K_D)$ can be rewritten as

$$\begin{aligned}
\mathcal{M}_1(K_D) &= -\frac{c^{1-\beta_1}K_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left[\frac{c}{X_F^*(K_D)}\left(\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right)^{\frac{1}{\beta_1}}\right]^{\beta_1}\\
&= -\frac{cK_D}{2(\beta_1-\beta_2)X_F^{*\beta_1}(K_D)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\\
&= -\frac{K_D}{X_F^{*\beta_1}(K_D)}\frac{\delta}{(1+\beta_1)F(\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right).
\end{aligned}$$

With $dK_F^*(K_D)/dK_D$ and $dX_F^*(K_D)/dK_D$ given by (3.47) and (3.48), it can be calculated that

$$\begin{aligned}
\frac{d\mathcal{M}_1(K_D)}{dK_D} &= -\frac{1-\gamma K_D-\beta_1\gamma K_D}{1-\gamma K_D}\left(\frac{1}{X_F^*(K_D)}\right)^{\beta_1}\frac{\delta}{F(\beta_2)(1+\beta_1)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\\
&= -\frac{c(1-\gamma K_D-\beta_1\gamma K_D)}{(1-\gamma K_D)c^{\beta_1}}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\frac{1}{2(\beta_1-\beta_2)}\left(\frac{c}{X_F^*}\right)^{\beta_1}\frac{2(\beta_1-\beta_2)\delta}{c(1+\beta_1)F(\beta_2)}\\
&= -\frac{c(1-\gamma K_D-\beta_1\gamma K_D)}{(1-\gamma K_D)c^{\beta_1}}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\frac{1}{2(\beta_1-\beta_2)}(1-\gamma K_D-2\gamma K_F^*)^{\beta_1}\\
&= \frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\left[-\frac{c^{1-\beta_1}K_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)(1-\gamma K_D-2\gamma K_F^*)^{\beta_1}\right]\\
&= \frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D).
\end{aligned}$$

Furthermore, it follows that

$$\begin{aligned}
&\frac{d}{dK_D}\mathcal{M}_2(K_D)X_F^{*\beta_2-\beta_1}(K_D)\\
&= \frac{c^{1-\beta_2}X_F^{*\beta_2-\beta_1}(K_D)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[(1-\gamma K_D)^{\beta_2}-\beta_2\gamma K_D(1-\gamma K_D)^{\beta_2-1}\right]\\
&\quad+\mathcal{M}_2(K_D)\frac{\gamma(\beta_2-\beta_1)X_F^{*\beta_2-\beta_1}(K_D)}{1-\gamma K_D}\\
&= \frac{(1-\gamma K_D-\beta_2\gamma K_D)X_F^{*\beta_2-\beta_1}(K_D)}{2(\beta_1-\beta_2)}\left(\frac{c}{1-\gamma K_D}\right)^{1-\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\\
&\quad+\mathcal{M}_2(K_D)\frac{\gamma(\beta_2-\beta_1)X_F^{*\beta_2-\beta_1}(K_D)}{1-\gamma K_D}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{(1-\gamma K_D-\beta_2\gamma K_D)X_F^{*\beta_2-\beta_1}(K_D)}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)+\frac{(\beta_2-\beta_1)\gamma K_D X_F^{*\beta_2-\beta_1}(K_D)}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)\\
=&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)X_F^{*\beta_2-\beta_1}(K_D).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\frac{d}{dK_D}\frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X_F^{*1-\beta_1}(K_D)\\
=&\frac{1}{2(r-\alpha)}\left[(1-2\gamma K_D)X_F^{*1-\beta_1}(K_D)+K_D(1-\gamma K_D)\frac{(1-\beta_1)\gamma X_F^{*1-\beta_1}(K_D)}{1-\gamma K_D}\right]\\
=&\frac{(1-\gamma K_D-\beta_1\gamma K_D)X_F^{*1-\beta_1}(K_D)}{2(r-\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dK_D}\frac{cK_D}{2r}X_F^{*-\beta_1}(K_D)&=\frac{c}{2r}\left(X_F^{*-\beta_1}(K_D)-\beta_1 K_D\frac{\gamma X_F^{*-\beta_1}(K_D)}{1-\gamma K_D}\right)\\
&=\frac{cX_F^{*-\beta_1}(K_D)(1-\gamma K_D-\beta_1\gamma K_D)}{2r(1-\gamma K_D)}.
\end{aligned}
$$

Thus, according to (3.16), we get that

$$
\begin{aligned}
\frac{d\mathcal{B}_1(K_D)}{dK_D}=&\frac{d\mathcal{M}_1(K_D)}{dK_D}+\frac{d}{dK_D}\mathcal{M}_2(K_D)X_F^{*\beta_2-\beta_1}(K_D)\\
&-\frac{d}{dK_D}\frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X_F^{*1-\beta_1}(K_D)+\frac{d}{dK_D}\frac{cK_D}{2r}X_F^{*-\beta_1}(K_D)\\
=&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)+\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)X_F^{*\beta_2-\beta_1}(K_D)\\
&-\frac{(1-\gamma K_D-\beta_1\gamma K_D)X_F^{*1-\beta_1}(K_D)}{2(r-\alpha)}+\frac{c(1-\gamma K_D-\beta_1\gamma K_D)X_F^{*-\beta_1}(K_D)}{2r(1-\gamma K_D)}\\
=&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_1(K_D).
\end{aligned}
$$

Next, we analyse the entry deterrence and accommodation strategies for the dedicated leader, which include the optimal investment capacities and optimal investment thresholds.

1. Entry Deterrence Strategy

The investment capacity $K_D^{det}(X)$ for a given level of X satisfies

$$
\begin{aligned}
\frac{\partial V_D(X,K_D)-\delta K_D}{\partial K_D} &= \frac{d\mathcal{B}_1(K_D)}{dK_D}X^{\beta_1}+\frac{1-2\gamma K_D}{r-\alpha}X-\frac{c}{r}-\delta \\
&= \frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_1(K_D)X^{\beta_1}+\frac{1-2\gamma K_D}{r-\alpha}X-\frac{c}{r}-\delta=0.
\end{aligned}
\tag{3.68}
$$

The entry deterrence strategy cannot happen when $K_D^{det}(X)<\hat{K}_D(X)$, which yields $X>X_2^{det}$ with X_2^{det} and $K_D^{det}(X_2^{det})$ satisfying (3.13) and (3.68). This is because the demand is high enough for the follower to invest immediately to enter the market. The entry deterrence strategy also does not happen when $K_D^{det}(X)<0$, yielding $X<X_1^{det}$ with X_1^{det} satisfying

$$
\begin{aligned}
\psi(X_1^{det}) = \Bigg[&-\frac{\delta}{(1+\beta_1)F(\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)+\frac{c^{1-\beta_2}X_F^{*\beta_2}(0)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right) \\
&-\frac{X_F^*(0)}{2(r-\alpha)}+\frac{c}{2r}\Bigg]\left(\frac{X_1^{det}}{X_F^*(0)}\right)^{\beta_1}+\frac{X_1^{det}}{r-\alpha}-\frac{c}{r}-\delta=0.
\end{aligned}
\tag{3.69}
$$

Thus, the entry deterrence strategy is only possible when $X\in(X_1^{det},X_2^{det})$. Suppose the investment threshold of the dedicated leader is $X^{det}(K_D)$ if the follower invests with capacity K_D in the entry deterrence strategy. The leader's value function before and after the investment is as follows

$$
V_D(X,K_D)=\begin{cases}
\mathcal{A}(K_D)X^{\beta_1} & X<X^{det}(K_D), \\
\mathcal{B}_1(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r} & X^{det}(K_D)\le X<X_F^*(K_D), \\
\mathcal{M}_1(K_D)X^{\beta_1}+\mathcal{M}_2(K_D)X^{\beta_2} & \\
+\frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X-\frac{cK_D}{2r} & X\ge X_F^*(K_D).
\end{cases}
\tag{3.70}
$$

The value matching and smooth pasting conditions to determine $X^{det}(K_D)$ are

$$
\begin{aligned}
\mathcal{A}(K_D)X^{\beta_1} &= \mathcal{B}_1(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r}-\delta K_D, \\
\beta_1\mathcal{A}(K_D)X^{\beta_1-1} &= \beta_1\mathcal{B}(K_D)X^{\beta_1-1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}.
\end{aligned}
$$

Thus, the threshold of the entry deterrence strategy $X^{det}(K_D)$ is

$$X^{det}(K_D) = \frac{\beta_1}{\beta_1 - 1}\frac{r-\alpha}{1-\gamma K_D}\left(\frac{c}{r}+\delta\right). \tag{3.71}$$

Substituting $X^{det}(K_D)$ into (3.68), the optimal investment capacity K_D^{det} and investment threshold $X^{det}(K_D^{det})$ can be derived as

$$\begin{aligned} K_D^{det} &\equiv K_D^{det}(X^{det}(K_D^{det})) = \frac{1}{(\beta_1+1)\gamma}, \\ X^{det}(K_D^{det}) &= \frac{(\beta_1+1)(r-\alpha)}{\beta_1 - 1}\left(\frac{c}{r}+\delta\right). \end{aligned}$$

2. Entry Accommodation Strategy
Note that from (3.62), we can get

$$\begin{aligned} \frac{\partial \mathcal{M}_2(K_D)}{\partial K_D} =& \frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2} \\ &+\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2-1}\frac{-c\beta_2\gamma K_D}{(1-\gamma K_D)^2} \\ =& \frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2}\left(1-\frac{\beta_2\gamma K_D}{1-\gamma K_D}\right) \\ =& \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2}\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)} \\ =& \frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D). \end{aligned}$$

The optimal capacity $K_D^{acc}(X)$ satisfies the first order condition

$$\begin{aligned} &\frac{\partial V(X,K_D)-\delta K_D}{\partial K_D} \\ =&\frac{d\mathcal{M}_1(K_D)}{dK_D}X^{\beta_1}+\frac{d\mathcal{M}_2(K_D)}{dK_D}X^{\beta_2}+\frac{1-2\gamma K_D}{2(r-\alpha)}X-\frac{c}{2r}-\delta \\ =&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)X^{\beta_1}+\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)X^{\beta_2} \\ &+\frac{1-2\gamma K_D}{2(r-\alpha)}X-\frac{c}{2r}-\delta = 0. \end{aligned} \tag{3.72}$$

The entry accommodation strategy only happens when $X \geq X_F^*(K_D)$, implying that the market demand is large enough to allow both the dedicated leader and the flexible follower to invest at the same time. Let X_1^{acc} be such that $X_1^{acc} = X_F^*(K_D^{acc}(X_1^{acc}))$, then X_1^{acc} and the corresponding $K_D^{acc}(X_1^{acc})$ satisfy (3.72) and (3.13). Suppose the dedicated leader invests at $X^{acc}(K_D)$ when the capacity level is K_D in the entry accommodation strategy, then the leader's value function before and after investment is

$$V_D(X,K_D) = \begin{cases} \mathcal{A}(K_D)X^{\beta_1} & X < X^{acc}(K_D), \\ \mathcal{M}_1(K_D)X^{\beta_1} + \mathcal{M}_2(K_D)X^{\beta_2} & \\ +\frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X - \frac{cK_D}{2r} & X \geq X_F^*(K_D) \geq X^{acc}(K_D). \end{cases} \tag{3.73}$$

The value matching and smooth pasting conditions to determine $X^{acc}(K_D)$ are

$$\mathcal{A}(K_D)X^{\beta_1} = \mathcal{M}_1(K_D)X^{\beta_1} + \mathcal{M}_2(K_D)X^{\beta_2} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}X - \frac{cK_D}{2r} - \delta K_D,$$
$$\beta_1\mathcal{A}(K_D)X^{\beta_1-1} = \beta_1\mathcal{M}_1(K_D)X^{\beta_1-1} + \beta_2\mathcal{M}_2(K_D)X^{\beta_2-1} + \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}.$$

Thus, the investment capacity $K_D^{acc}(X^{acc})$ and investment threshold $X^{acc}(K_D^{acc})$ satisfy equation (3.72) and

$$(\beta_1-\beta_2)\mathcal{M}_2(K_D)X^{\beta_2} + \frac{(\beta_1-1)K_D(1-\gamma K_D)}{2(r-\alpha)}X - \frac{c\beta_1 K_D}{2r} - \beta_1\delta K_D = 0. \tag{3.74}$$

Rewrite these two equations, then

$$\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)X^{\beta_2}$$
$$+\frac{1-\gamma K_D-\beta_2\gamma K_D}{1-\gamma K_D}\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X}{X_1}\right)^{\beta_2} + \frac{1-2\gamma K_D}{2(r-\alpha)}X - \frac{c}{2r} - \delta = 0,$$

$$\frac{c}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X}{X_1}\right)^{\beta_2} + \frac{(\beta_1-1)(1-\gamma K_D)}{2\beta_1(r-\alpha)}X - \frac{c}{2r} - \delta = 0.$$

Solving these equations, we can get

$$K_D^{acc} \equiv K_D^{acc}(X^{acc}(K_D^{acc})) = \frac{1}{(\beta_1+1)\gamma}.$$

Let

$$Z(X) = \frac{c}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\left(\frac{\beta_1}{(\beta_1+1)c}\right)^{\beta_2} X^{\beta_2} + \frac{\beta_1-1}{2(\beta_1+1)(r-\alpha)}X - \frac{c}{2r} - \delta.$$

Then

$$\begin{aligned}
&Z\left(\frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right)\right)\\
=&\frac{c}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\left(\frac{\beta_1(r-\alpha)}{(\beta_1-1)c}\left(\frac{c}{r}+\delta\right)\right)^{\beta_2} + \frac{1}{2}\left(\frac{c}{r}+\delta\right) - \frac{c}{2r} - \delta\\
=&\frac{c}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\left(\frac{\beta_1(r-\alpha)}{(\beta_1-1)c}\left(\frac{c}{r}+\delta\right)\right)^{\beta_2} - \frac{\delta}{2} < 0.
\end{aligned}$$

Because

$$\frac{dZ(X)}{dX} = \frac{c\beta_2}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\left(\frac{\beta_1}{(\beta_1+1)c}\right)^{\beta_2} X^{\beta_2-1} + \frac{\beta_1-1}{2(\beta_1+1)(r-\alpha)} > 0,$$

it can be concluded that

$$X^{acc}(K_D^{acc}) > \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right).$$

Proof of Negative $\mathcal{B}_2(K_D)$ When the flexible follower produces up to capacity right after investment, we have

$$\begin{aligned}
\mathcal{B}_2(K_D) =&\mathcal{N}(K_D)X_F^{*\beta_2-\beta_1}(K_D) - \frac{\gamma K_D K_F^*(K_D)}{r-\alpha}X_F^{*1-\beta_1}(K_D)\\
=&\frac{c^{1-\beta_2}K_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha} - \frac{\beta_1}{r}\right)\cdot\\
&\left[(1-\gamma K_D)^{\beta_2} - (1-\gamma K_D - 2\gamma K_F(K_D))^{\beta_2}\right] X_F^{*\beta_2-\beta_1}(K_D)\\
&-\frac{\gamma K_D K_F^*(K_D)}{r-\alpha}X_F^{*1-\beta_1}(K_D)
\end{aligned}$$

$$=\frac{cK_DX_F^{*-\beta_1}(K_D)}{2(\beta_1-\beta_2)}\left\{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{X_F^*(K_D)}{X_1(K_D)}\right)^{\beta_2}-\left(\frac{X_F^*(K_D)}{X_2(K_D)}\right)^{\beta_2}\right)\right.$$
$$\left.+\frac{\beta_1-\beta_2}{r-\alpha}\left(\frac{X_F^*(K_D)}{X_2(K_D)}-\frac{X_F^*(K_D)}{X_1(K_D)}\right)\right\}.$$

Note that

$$\begin{aligned}\frac{dX_F^*(K_D)}{dK_D} &= \frac{\gamma X_F^*(K_D)}{1-\gamma K_D},\\ \frac{dX_1(K_D)}{dK_D} &= \frac{\gamma X_1(K_D)}{1-\gamma K_D},\\ \frac{dX_2(K_D)}{dK_D} &= \frac{\gamma X_2(K_D)}{1-\gamma K_D}.\end{aligned}$$

Thus, for the terms in $\mathcal{B}_2(K_D)$, we have

$$\begin{aligned}\frac{d}{dK_D}\frac{X_F^*(K_D)}{X_1(K_D)} &= \frac{1}{X_1^2(K_D)}\left(\frac{\gamma X_F^*(K_D)X_1(K_D)}{1-\gamma K_D}-\frac{\gamma X_F^*(K_D)X_1(K_D)}{1-\gamma K_D}\right)=0,\\ \frac{d}{dK_D}\frac{X_F^*(K_D)}{X_2(K_D)} &= \frac{1}{X_2^2(K_D)}\left(\frac{\gamma X_F^*(K_D)X_2(K_D)}{1-\gamma K_D}-\frac{\gamma X_F^*(K_D)X_2(K_D)}{1-\gamma K_D}\right)=0.\end{aligned}$$

Similar to the case that flexible follower produces below capacity right after investment, $X_F^*(K_D)/X_1(K_D)$ and $X_F^*(K_D)/X_2(K_D)$ are constants and do not change with K_D. Thus

$$\begin{aligned}\frac{X_F^*(K_D)}{X_1(K_D)} &= \frac{X_F^*(0)}{c},\\ \frac{X_F^*(K_D)}{X_2(K_D)} &= \frac{X_F^*(0)}{X_2(0)}.\end{aligned}$$

Let $X_F^*(0)=X^*$ and $X_2(0)=1-2\gamma K^*$, with X^* as the optimal investment threshold and K^* as the optimal capacity in the monopoly case where the firm produces up to capacity right after investment. We rewrite $\mathcal{B}_2(K_D)$ as

$$\mathcal{B}_2(K_D)=\frac{cK_DX_F^{*-\beta_1}(K_D)}{2(\beta_1-\beta_2)}\mathcal{G}(X^*,K^*),$$

where X^* and K^* satisfy

$$\frac{c(1+\beta_2)F(\beta_1)}{2(\beta_1-\beta_2)}\left(\frac{(1-2\gamma K^*)X^*}{c}\right)^{\beta_2}+\frac{c}{r-\alpha}\frac{(1-2\gamma K^*)X^*}{c}-\frac{c}{r}-\delta=0$$

and

$$\begin{aligned}&\frac{cF(\beta_1)}{4\gamma\beta_1}\left(\left(\frac{X^*}{c}\right)^{\beta_2}-(1-2\gamma K^*)\left(\frac{(1-2\gamma K^*)X^*}{c}\right)^{\beta_2}\right)\\&+\frac{\beta_1-1}{\beta_1}\frac{(1-\gamma K^*)X^*K^*}{r-\alpha}-\frac{cK^*}{r}-\delta K^*=0.\end{aligned}$$

$\mathcal{B}_2(K_D)$ is intuitively negative. However, it is too complicated to show this analytically. So we try to show it is negative numerically to verify the conjecture. Figure 3.21 demonstrates $\mathcal{G}(X^*,K^*)$ changing with parameters. The default parameter values are given as $\alpha=0.02$, $r=0.1$, $\sigma=0.2$, $c=2$, $\delta=10$, and $\gamma=0.05$. Some combination of parameter values does not define the case that the follower produces up to capacity right after investment. After ruling out such combinations, the negative $\mathcal{G}(X^*,K^*)$ is illustrated in Figure 3.21. This confirms the conjecture that $\mathcal{B}_2(K_D)$ is negative. So in the following analysis, we assume negative $\mathcal{B}_2(K_D)$.

Proof of Proposition 3.3 We start with the derivative of $\mathcal{B}_2(K_D)$ with respect to K_D, where $K_F^*(K_D)$ and $X_F^*(K_D)$ are defined by (3.11) and (3.12), and $dK_F^*(K_D)/dK_D$ and $dX_F^*(K_D)/dK_D$ are defined by (3.55) and (3.56). It holds that

$$\begin{aligned}\frac{d\mathcal{N}(K_D)}{dK_D}=&\frac{c^{1-\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\Big[(1-\gamma K_D)^{\beta_2}-(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}\\&-\gamma\beta_2K_D\left((1-\gamma K_D)^{\beta_2-1}-\left(1+2\frac{dK_F^*(K_D)}{dK_D}\right)(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2-1}\right)\Big]\\=&\frac{c^{1-\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\Big[(1-\gamma K_D)^{\beta_2}-(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}\\&-\gamma\beta_2K_D\left((1-\gamma K_D)^{\beta_2-1}-\frac{(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}}{1-\gamma K_D}\right)\Big]\\=&\frac{c^{1-\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\Big[(1-\gamma K_D)^{\beta_2}-(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}\Big]\cdot\\&\frac{1-\gamma K_D-\beta_2\gamma K_D}{1-\gamma K_D}\\=&\frac{X_1(1-\gamma K_D-\beta_2\gamma K_D)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(X_1^{-\beta_2}-X_2^{-\beta_2}\right)\end{aligned}$$

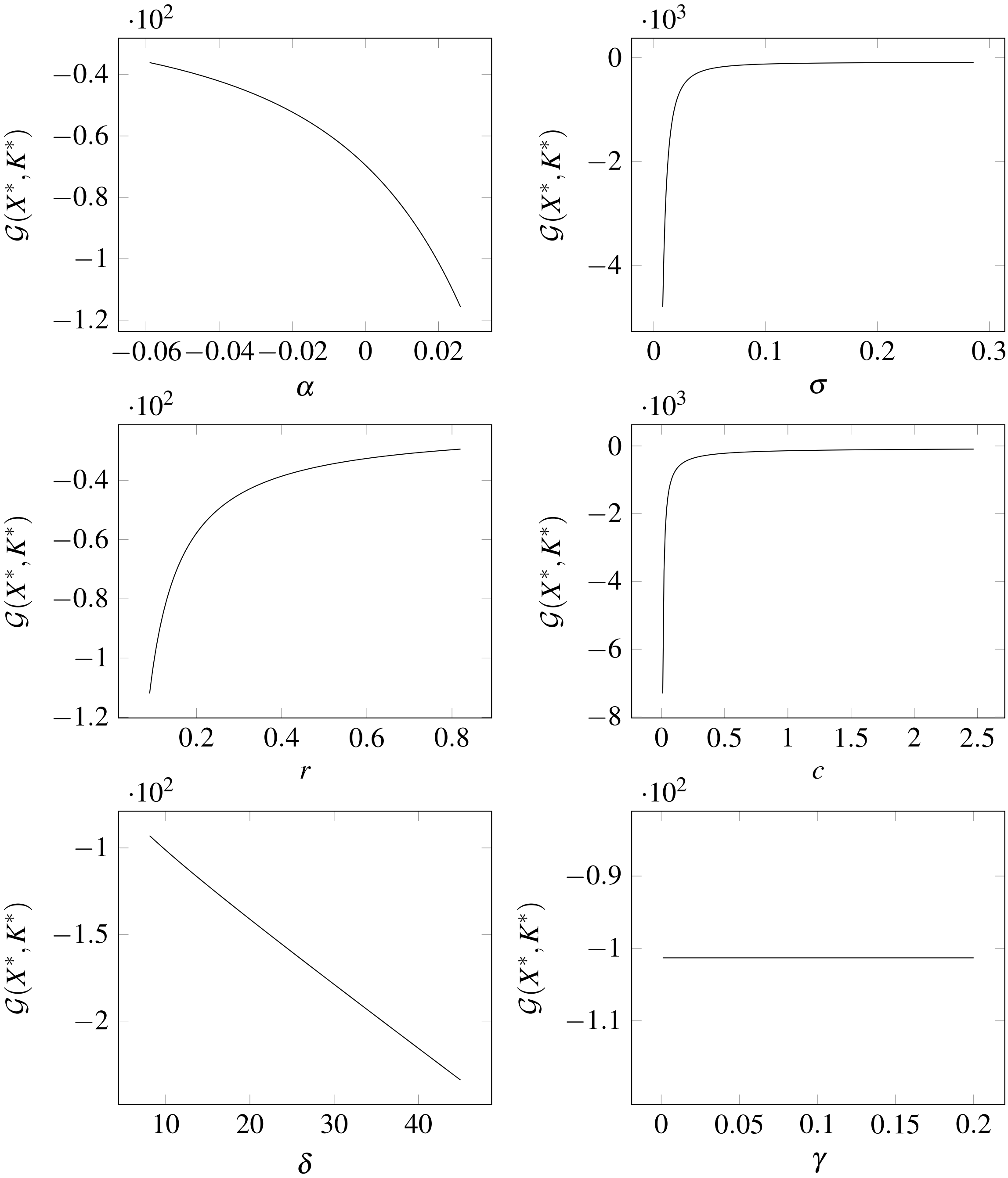


Figure 3.21: Illustration of negative $\mathcal{G}(X^*, K^*)$ changing with α, σ, r, c, δ, and γ. Default parameter values are $\alpha = 0.02$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$, $\gamma = 0.05$.

$$=\frac{\mathcal{N}(K_D)(1-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)},$$

$$\begin{aligned}\frac{d\mathcal{N}(K_D)X_F^{*\beta_2-\beta_1}}{dK_D} =&\frac{d\mathcal{N}(K_D)}{dK_D}X_F^{*\beta_2-\beta_1}+\mathcal{N}(K_D)X_F^{*\beta_2-\beta_1}\frac{\gamma(\beta_2-\beta_1)}{1-\gamma K_D}\\ =&\frac{(1-\gamma K_D-\beta_2\gamma K_D)\mathcal{N}(K_D)}{K_D(1-\gamma K_D)}X_F^{*\beta_2-\beta_1}+\frac{(\beta_2\gamma K_D-\beta_1\gamma K_D)\mathcal{N}(K_D)}{K_D(1-\gamma K_D)}X_F^{*\beta_2-\beta_1}\\ =&\frac{(1-\gamma K_D-\beta_1\gamma K_D)\mathcal{N}(K_D)}{K_D(1-\gamma K_D)}X_F^{*\beta_2-\beta_1},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dK_D}K_DK_F^*X_F^{*1-\beta_1} =&K_F^*X_F^{*1-\beta_1}+K_D\frac{-\gamma K_F^*}{1-\gamma K_D}X_F^{*1-\beta_1}+K_DK_F^*(1-\beta_1)X_F^{*-\beta_1}\frac{\gamma X_F^*}{1-\gamma K_D}\\ =&K_F^*X_F^{*1-\beta_1}-\frac{\gamma K_DK_F^*X_F^{*1-\beta_1}}{1-\gamma K_D}+\frac{\gamma(1-\beta_1)K_DK_F^*X_F^{*1-\beta_1}}{1-\gamma K_D}\\ =&K_F^*X_F^{*1-\beta_1}-\frac{\beta_1\gamma}{1-\gamma K_D}K_DK_F^*X_F^{*1-\beta_1}\\ =&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}K_DK_F^*X_F^{*1-\beta_1}.\end{aligned}$$

Thus,

$$\frac{d\mathcal{B}_2(K_D)}{dK_D}=\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_2(K_D).$$

1. Entry Deterrence Strategy
   The optimal capacity by the dedicated leader, $K_D^{det}(X)$, satisfies the first order condition

$$\begin{aligned}\frac{\partial V_D(X,K_D)-\delta K_D}{\partial K_D} =&\frac{d\mathcal{B}_2(K_D)}{dK_D}X^{\beta_1}+\frac{1-2\gamma K_D}{r-\alpha}X-\frac{c}{r}-\delta\\ =&\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_2(K_D)X^{\beta_1}+\frac{1-2\gamma K_D}{r-\alpha}X-\frac{c}{r}-\delta=0.\end{aligned} \tag{3.75}$$

   The entry deterrence strategy cannot happen if $K_D^{det}(X)<\hat{K}_D(X)$. If we assume that the dedicated leader invests at X, then the deterrence strategy is only possible when $X<X_2^{det}$. X_2^{det}, $K_D^{det}(X_2^{det})$, and $K_F^*(K_D^{det})$ satisfy (3.11), (3.12), and (3.75), with $X_F^*(K_D^{det})=X_2^{det}$. Similar to the case that the flexible follower produces below capacity right after investment, the deterrence strategy is not possible if $K_D^{det}<0$, which results that $X > X_1^{det}$ with X_1^{det} satisfying

$$\frac{c}{2(\beta_1-\beta_2)}\left(\frac{X_1^{det}}{X_F^*(0)}\right)^{\beta_1}\left\{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[\left(\frac{X_F^*(0)}{c}\right)^{\beta_2}-\left(\frac{X_F^*(0)(1-2\gamma K_F^*(0))}{c}\right)^{\beta_2}\right]\right.$$
$$\left.-\frac{\beta_1-\beta_2}{r-\alpha}\frac{2\gamma X_F^*(0)K_F^*(0)}{c}\right\}+\frac{1}{r-\alpha}-\frac{c}{r}-\delta=0, \tag{3.76}$$

where $K_F^*(0)$ and $X_F^*(0)$ satisfy (3.11) and (3.12). Thus, the entry deterrence strategy is only possible if $X \in \left(X_1^{det}, X_2^{det}\right)$. If the leader applies the entry deterrence strategy and invests at $X^{det}(K_D)$ with capacity level K_D, then the value function before and after investment is

$$V_D(X,K_D)=\begin{cases}\mathcal{A}(K_D)X^{\beta_1} & X < X^{det}(K_D),\\ \mathcal{B}_2(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r} & X^{det}(K_D)\le X < X_F^*(K_D),\\ \mathcal{N}(K_D)X^{\beta_2}+\frac{K_D(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}X-\frac{cK_D}{r} & X \ge X_F^*(K_D).\end{cases} \tag{3.77}$$

For a given capacity level K_D, from value matching and smooth pasting at $X^{det}(K_D)$, $X^{det}(K_D)$ must satisfy

$$\begin{aligned}\mathcal{A}(K_D)X^{\beta_1} &= \mathcal{B}_2(K_D)X^{\beta_1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}X-\frac{cK_D}{r}-\delta K_D,\\ \beta_1\mathcal{A}(K_D)X^{\beta_1-1} &= \beta_1\mathcal{B}_2(K_D)X^{\beta_1-1}+\frac{K_D(1-\gamma K_D)}{r-\alpha}.\end{aligned}$$

It can be derived that

$$X^{det}(K_D)=\frac{\beta_1(r-\alpha)}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right). \tag{3.78}$$

K_D^{det} and $X^{det}(K_D^{det})$ satisfy (3.75), thus the optimal investment capacity K_D^{det} and investment threshold $X^{det}(K_D^{det})$ are

$$\begin{aligned}K_D^{det} &\equiv K_D^{det}(X^{det}(K_D^{det}))=\frac{1}{(\beta_1+1)\gamma},\\ X^{det}(K_D^{det}) &= \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right).\end{aligned}$$

2. Entry Accommodation Strategy

We first derive the optimal capacity under the accommodation strategy. Note that

$$\mathcal{N}(K_D) = \frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[X_1^{-\beta_2}(K_D)-X_2^{-\beta_2}(K_D)\right],$$

with

$$\begin{aligned} X_1(K_D) &= \frac{c}{1-\gamma K_D}, \\ X_2(K_D) &= \frac{c}{1-\gamma K_D - 2\gamma K_F^*(K_D)}. \end{aligned}$$

Because

$$\frac{d}{dK_D}X_1^{-\beta_2}(K_D) = -\frac{\gamma\beta_2 X_1^{-\beta_2}(K_D)}{1-\gamma K_D}$$

and

$$\frac{d}{dK_D}X_2^{-\beta_2}(K_D) = -\frac{\gamma\beta_2 X_2^{-\beta_2}(K_D)}{1-\gamma K_D},$$

then it holds that

$$\begin{aligned} \frac{d\mathcal{N}(K_D)}{dK_D} =& \frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\Bigg[X_1^{-\beta_2}(K_D)-X_2^{-\beta_2}(K_D) \\ & -\frac{\gamma\beta_2 K_D}{1-\gamma K_D}\left(X_1^{-\beta_2}(K_D)-X_2^{-\beta_2}(K_D)\right)\Bigg] \\ =& \frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(X_1^{-\beta_2}(K_D)-X_2^{-\beta_2}(K_D)\right)\frac{1-\gamma K_D-\beta_2\gamma K_D}{1-\gamma K_D} \\ =& \frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{N}(K_D). \end{aligned}$$

Thus, the investment capacity by the dedicated leader $K_D^{acc}(X)$ for a given level of X satisfies the first order condition

$$\begin{aligned} \frac{\partial V_D(X,K_D)-\delta K_D}{\partial K_D} =& \frac{d\mathcal{N}(K_D)}{dK_D}X^{\beta_2}+\frac{X}{r-\alpha}\Bigg[(1-\gamma K_D-\gamma K_F^*(K_D)) \\ & -\gamma K_D\left(1+\frac{dK_F^*(K_D)}{dK_D}\right)\Bigg]-\frac{c}{r}-\delta \\ =& \frac{(1-\gamma K_D-\beta_2\gamma K_D)\mathcal{N}(K_D)}{K_D(1-\gamma K_D)}X^{\beta_2}-\frac{c}{r}-\delta \end{aligned}$$

$$+\frac{X(1-\gamma K_D-\gamma K_F^*(K_D))(1-2\gamma K_D)}{(r-\alpha)(1-\gamma K_D)}=0 \tag{3.79}$$

The entry accommodation strategy only happens when the market has grown large enough to hold the two firms, i.e., $X \geq X_F^*(K_D)$. Define $X_1^{acc} = X_F^*\left(K_D^{acc}(X_1^{acc})\right)$, then X_1^{acc}, $K_D^{acc}(X_1^{acc})$, and $K_F^*(K_D^{acc})$ satisfy (3.11), (3.12), and (3.79). Suppose the dedicated leader uses the entry accommodation strategy and invests at $X^{acc}(K_D)$ with capacity K_D, then the leader's value function is

$$V_D(X,K_D)=\begin{cases}\mathcal{A}(K_D)X^{\beta_1} & X<X^{acc}(K_D),\\ \mathcal{N}(K_D)X^{\beta_2}+\frac{K_D(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}X-\frac{cK_D}{r} & X\geq X_F^*(K_D)\geq X^{acc}(K_D).\end{cases} \tag{3.80}$$

From value matching and smooth pasting, we get that the investment threshold $X^{acc}(K_D)$ satisfies

$$\begin{aligned}\mathcal{A}(K_D)X^{\beta_1} &= \mathcal{N}(K_D)X^{\beta_2}+\frac{K_D(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}X-\frac{cK_D}{r}-\delta K_D,\\ \beta_1\mathcal{A}(K_D)X^{\beta_1-1} &= \beta_2\mathcal{N}(K_D)X^{\beta_2-1}+\frac{K_D(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}.\end{aligned}$$

Thus, it holds that $X^{acc}(K_D)$ must satisfy

$$\frac{\beta_1-\beta_2}{\beta_1}\mathcal{N}(K_D)X^{\beta_2}+\frac{\beta_1-1}{\beta_1(r-\alpha)}XK_D\left(1-\gamma K_D-\gamma K_F^*(K_D)\right)-\frac{cK_D}{r}-\delta K_D=0. \tag{3.81}$$

Rewrite (3.79) and (3.81), then $X^{acc}(K_D^{acc})$ and K_D^{acc} satisfy

$$\begin{aligned}&\frac{1-\gamma K_D-\beta_2\gamma K_D}{1-\gamma K_D}\frac{cX^{\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(X_1^{-\beta_2}-X_2^{-\beta_2}\right)\\ &\qquad+\frac{X(1-\gamma K_D-\gamma K_F^*(K_D))}{r-\alpha}\frac{1-2\gamma K_D}{1-\gamma K_D}-\frac{c}{r}-\delta=0,\\ &\frac{cX^{\beta_2}}{2\beta_1}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(X_1^{-\beta_2}-X_2^{-\beta_2}\right)+\frac{X(1-\gamma K_D-\gamma F^*(K_D))}{r-\alpha}\frac{\beta_1-1}{\beta_1}\\ &\qquad-\frac{c}{r}-\delta=0.\end{aligned}$$

From

$$\frac{1-\gamma K_D-\beta_2\gamma K_D}{(\beta_1-\beta_2)(1-\gamma K_D)}=\frac{1}{\beta_1},$$

and

$$\frac{1-2\gamma K_D}{1-\gamma K_D}=\frac{\beta_1-1}{\beta_1},$$

it follows that the optimal investment capacity is

$$K_D^{acc}\equiv K_D^{acc}(X^{acc}(K_D^{acc}))=\frac{1}{(\beta_1+1)\gamma}.$$

Proof of Proposition 3.4 Given in the text.
Proof of Proposition 3.5 When there is no flexibility, the leader's entry deterrence and entry accommodation strategy can be found in Appendix 3.7.2. When the follower is flexible, from Proposition 3.2 and Proposition 3.3, the leader's entry deterrence strategy is

$$\begin{aligned} X_D^{det} &= \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right), \\ K_D^{det} &= \frac{1}{(\beta+1)\gamma}, \end{aligned}$$

regardless of whether the follower produces below or up to capacity right after investment. This is the same as the the leader's entry deterrence strategy when there is no flexibility. From Proposition 3.2 and Proposition 3.3, it also holds that the leader's investment capacity under entry accommodation strategy is $K_D^{acc}=\frac{1}{(\beta+1)\gamma}$, regardless of whether the follower produces below or up to capacity right after investment. This capacity level is the same as that when there is no flexibility.

3.7.2 No Flexibility

This section analyzes what the follower and leader's decisions are when there is no flexibility. It means that both firms would always produce up to full capacity. For the follower, given that the leader invests and always produces K_D and the follower invests and always produces K_F, the profit flow at time t equals

$$\pi_F(t)=(X(t)\,(1-\gamma(K_D+K_F))-c)\,K_F.$$

Here, we do not allow production suspension. So for a low level X, i.e., $X\,(1-\gamma(K_D+K_F))<c$, the firms may have negative profit flows. Given the initial geometric Brownian motion

level X, the value of the follower is

$$\begin{aligned} V_F(X,K_D,K_F) =& E\left[\int_{t=0}^{\infty} K_F\left(X(t)\left(1-\gamma(K_D+K_F)\right)-c\right)\exp\left(-rt\right)dt \,|X(0)=X\right] \\ =& \frac{XK_F(1-\gamma(K_D+K_F))}{r-\alpha}-\frac{cK_F}{r}. \end{aligned}$$

The follower's investment capacity maximizes

$$\max_{K_F>0} V_F(X,K_D,K_F)-\delta K_F,$$

thus, given X and K_D,

$$K_F(X,K_D)=\frac{1}{2\gamma}\left(1-\gamma K_D-\frac{r-\alpha}{X}\left(\frac{c}{r}+\delta\right)\right). \tag{3.82}$$

Before the investment, the follower holds an option to invest. Suppose the option value is

$$V_F(X,K_D)=A_F(K_L)X^{\beta_1}.$$

According to value matching and smooth pasting, the investment threshold $X_F(K_D,K_F)$ when investing with K_F satisfies

$$\begin{aligned} A_F X_F^{\beta_1} &= \frac{X_F^* K_F(1-\gamma(K_D+K_F))}{r-\alpha}-\frac{cK_F}{r}-\delta K_F, \\ \beta_1 A_F X_F^{\beta_1-1} &= \frac{K_F(1-\gamma(K_D+K_F))}{r-\alpha}. \end{aligned}$$

Thus,

$$X_F(K_D,K_F)=\frac{\beta_1(r-\alpha)}{(\beta_1-1)(1-\gamma K_D-\gamma K_F)}\left(\frac{c}{r}+\delta\right). \tag{3.83}$$

Combining (3.82) and (3.83), the follower's optimal investment capacity and threshold are

$$K_F^*(K_D) = \frac{1-\gamma K_D}{(1+\beta_1)\gamma}, \tag{3.84}$$

$$X_F^*(K_D) = \frac{(\beta_1+1)(r-\alpha)}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right). \tag{3.85}$$

If $X_F^*(K_D) \le X(0)$, then the follower would invest immediately at $t = 0$ with capacity $K_F^*(X(0), K_D)$.

For the leader, to deter or accommodate the entry of the follower would be dependent on the leader's critical capacity level

$$\hat{K}_D(X) = \frac{1}{\gamma}\left(1 - \frac{(\beta_1+1)(r-\alpha)}{(\beta_1-1)X}\left(\frac{c}{r}+\delta\right)\right). \tag{3.86}$$

Entry Deterrence Strategy If the leader invests a capacity larger than $\hat{K}_D(X)$, then the follower invests later. However, if the leader invests a capacity not larger than $\hat{K}_D(X)$, then the follower invests at the same time with the leader. Suppose the investment threshold is $X_D^{det}(K_D)$ when investing capacity K_D, then the leader's value under entry deterrence strategy is assumed to be

$$V_D(X,K_D) = \begin{cases} A_D(K_D)X^{\beta_1} & \text{if } X < X_D^{det}(K_D), \\ B_D(K_D)X^{\beta_1} + \frac{XK_D(1-\gamma K_D)}{r-\alpha} - \frac{cK_D}{r} & \text{if } X_D^{\text{det}}(K_D) \le X < X_F^*(K_D), \\ \frac{\beta_1 XK_D(1-\gamma K_D)}{(1+\beta_1)(1-\alpha)} - \frac{cK_D}{r} & \text{if } X \ge X_F^*(K_D). \end{cases}$$

By value matching at $X_F^*(K_D)$, we get

$$B_D(K_D)X_F^{*\beta_1} + \frac{X_F^* K_D(1-\gamma K_D)}{r-\alpha} = \frac{\beta_1 X_F^* K_D(1-\gamma K_D)}{(\beta_1+1)(r-\alpha)}.$$

Thus,

$$\begin{aligned} B_D(K_D) &= -\frac{K_D(1-\gamma K_D)X_F^*}{(\beta_1+1)(r-\alpha)} X_F^{*-\beta_1} \\ &= -\frac{K_D}{\beta_1-1}\left(\frac{c}{r}+\delta\right)\left(\frac{(\beta_1+1)(r-\alpha)}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right)\right)^{-\beta_1}. \end{aligned}$$

Suppose the leader invests at X, then the investment capacity under the deterrence strategy, $K_D^{det}(X)$, satisfies

$$-\frac{1-(\beta_1+1)\gamma K_D}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right)\left(\frac{X(\beta_1-1)(1-\gamma K_D)}{(\beta_1+1)(r-\alpha)\left(\frac{c}{r}+\delta\right)}\right)^{\beta_1} + \frac{X(1-2\gamma K_D)}{r-\alpha} - \frac{c}{r} - \delta = 0. \tag{3.87}$$

The corresponding value for the leader's entry deterrence strategy is

$$V_D^{det}(X) = -\frac{K_D^{det}(X)}{\beta_1 - 1}\left(\frac{c}{r}+\delta\right)\left(\frac{X(\beta_1-1)(1-\gamma K_D^{det}(X))}{(\beta_1+1)(r-\alpha)\left(\frac{c}{r}+\delta\right)}\right)^{\beta_1} + \frac{XK_D^{det}(X)(1-\gamma K_D^{det}(X))}{r-\alpha} - \frac{cK_D^{det}(X)}{r} - \delta K_D^{det}(X). \tag{3.88}$$

If X is sufficiently small, then the optimal investment threshold is

$$X_D^{det} = \frac{\beta_1(r-\alpha)}{(\beta_1-1)(1-\gamma K_D^{det})}\left(\frac{c}{r}+\delta\right). \tag{3.89}$$

Substitute (3.89) into (3.87) gives

$$1-(\beta_1+1)\gamma K_D^{det} = \left(1-(\beta_1+1)\gamma K_D^{det}\right)\left(\frac{\beta_1}{\beta_1+1}\right)^{\beta_1}.$$

Thus, we have

$$\begin{aligned} K_D^{det} &= \frac{1}{(\beta_1+1)\gamma}, \\ X_D^{det} &\equiv X_D^{det}(K_D^{det}) = \frac{(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right). \end{aligned}$$

The corresponding follower's investment decisions are

$$\begin{aligned} K_F^*(K_D^{det}) &= \frac{\beta_1}{(\beta_1+1)^2\gamma}, \\ X_F^*(K_D^{det}) &= \frac{(\beta_1+1)^2(r-\alpha)}{\beta_1(\beta_1-1)}\left(\frac{c}{r}+\delta\right). \end{aligned}$$

Moreover, the entry deterrence strategy can not happen for

$$0 \leq \hat{K}_D(X) < K_D^{det},$$

i.e.,

$$X_1^{det} \leq X \leq X_2^{det},$$

where

$$X_2^{det} = \frac{2(\beta_1+1)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right)$$

and X_1^{det} satisfies

$$-\frac{1}{\beta_1-1}\left(\frac{c}{r}+\delta\right)\left(\frac{X(\beta_1-1)}{(\beta_1+1)(r-\alpha)\left(\frac{c}{r}+\delta\right)}\right)^{\beta_1}+\frac{X}{r-\alpha}-\frac{c}{r}-\delta=0. \tag{3.90}$$

If $X_D^{det} \le X$, then the deterrence strategy is implemented immediately with capacity $K_D^{det}(X)$ satisfying (3.87).

Entry Accommodation Strategy Under the entry accommodation strategy, the follower invests at the same time as the leader. Suppose the investment threshold is $X_D^{acc}(K_D)$ when investing capacity K_D, then the leader's value under entry accommodation strategy is assumed to be

$$V_D(X,K_D) = \begin{cases} A_D(K_D)X^{\beta_1} & \text{if } X < X_D^{acc}(K_D), \\ \frac{XK_D(1-\gamma K_D)}{2(r-\alpha)} - \frac{cK_D}{2r} + \frac{\delta K_D}{2} & \text{if } X \ge X_D^{acc}(K_D). \end{cases}$$

For a given level of X, the investment capacity under the entry accommodation strategy is

$$K_D^{acc}(X) = \frac{1}{2\gamma}\left(1-\frac{r-\alpha}{X}\left(\frac{c}{r}+\delta\right)\right).$$

The accommodation strategy can only be chosen when $K_D^{acc}(X) \le \hat{K}_D(X)$, which means that it is only possible when

$$X \ge X_1^{acc} = \frac{(\beta_1+3)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right).$$

Moreover, the value matching and smoothing pasting conditions yield that for the given capacity K_D, the investment threshold $X_D^{acc}(K_D)$ satisfies

$$\begin{aligned} A_D(K_D)X^{\beta_1} &= \frac{XK_D(1-\gamma K_D)}{2(r-\alpha)} - \frac{cK_D}{2r} - \frac{\delta K_D}{2}, \\ \beta_1 A_D(K_D)X^{\beta-1} &= \frac{K_D(1-\gamma K_D)}{2(r-\alpha)}. \end{aligned}$$

Thus, it holds that

$$X_D^{acc}(K_D) = \frac{\beta_1(r-\alpha)}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right).$$

Then the optimal investment capacity K_D^{acc} and the optimal investment threshold X_D^{acc} are

$$\begin{aligned} K_D^{acc} &= \frac{1}{(\beta_1+1)\gamma}, \\ X_D^{acc} &= \frac{(\beta_1+1)(r-\alpha)}{(\beta_1-1)}\left(\frac{c}{r}+\delta\right). \end{aligned}$$

If $X_D^{acc} \leq X$, then the leader invests immediately at X with capacity

$$K_D(X) = \frac{1}{2\gamma}\left(1-\frac{r-\alpha}{X}\left(\frac{c}{r}+\delta\right)\right).$$

Note that $X_1^{acc} > X_D^{acc}$. This means that the leader implements the accommodation strategy only when X reaches X_1^{acc}. Then the leader invests at X_1^{acc} with capacity

$$K_D(X_1^{acc}) = \frac{2}{(\beta_1+3)\gamma}.$$

The leader's value at X_1^{acc} is

$$V_D(X_1^{acc}, K_D(X_1^{acc})) = \frac{2}{(\beta_1-1)(\beta_1+3)\gamma}\left(\frac{c}{r}+\delta\right).$$

The corresponding follower's investment decisions under the leader's accommodation strategy are

$$\begin{aligned} K_F^*(K_1^{acc}) &= \frac{\beta_1+1}{(\beta_1+3)\gamma}, \\ X_F^*(K_1^{acc}) &= \frac{(\beta_1+3)(r-\alpha)}{\beta_1-1}\left(\frac{c}{r}+\delta\right). \end{aligned}$$

3.7.3 Additional Proof: Negative Second Order Derivatives

Producing below capacity right after investment

Proof of negative second order partial derivative under the entry deterrence strategy

Under the entry deterrence strategy when the follower produces below capacity right after investment, the second order partial derivative of $V(X,K_D)-\delta K_D$ with respect to K_D is

$$
\begin{aligned}
&\frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2}\\
=&\frac{(-\gamma-\beta_1\gamma)K_D(1-\gamma K_D)-(1-\gamma K_D-\beta_1\gamma K_D)(1-2\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}\\
&+\frac{(1-\gamma K_D-\beta_1\gamma K_D)^2}{K_D^2(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
=&\frac{-\gamma K_D(1+\beta_1)(1-\gamma K_D)+\gamma K_D(1-\gamma K_D-\beta_1\gamma K_D)(1-\beta_1)}{K_D^2(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
=&\frac{-\gamma(1+\beta_1)(1-\gamma K_D)+\gamma(1-\beta_1)(1-\gamma K_D)-\beta_1\gamma K_D(1-\beta_1)\gamma}{K_D(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
=&\frac{-2\beta_1\gamma(1-\gamma K_D)-\beta_1\gamma(1-\beta_1)\gamma K_D}{K_D(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
=&\frac{-\beta_1\gamma(2-\gamma K_D-\beta_1\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{B}_1(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}.
\end{aligned}
$$

The third order derivative with respect to K_D is

$$
\begin{aligned}
&\frac{\partial^3[V(X,K_D)-\delta K_D]}{\partial K_D^3}\\
=&-\frac{\beta_1\gamma\big[(-\gamma-\beta_1\gamma)K_D(1-\gamma K_D)^2-(2-\gamma K_D-\beta_1\gamma K_D)(1-\gamma K_D)(1-3\gamma K_D)\big]}{K_D^2(1-\gamma K_D)^4}\mathcal{B}_1(K_D)X^{\beta_1}\\
&-\frac{\beta_1\gamma(2-\gamma K_D-\beta_1\gamma K_D)}{K_D(1-\gamma K_D)^2}\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{B}_1(K_D)X^{\beta_1}\\
=&\frac{-\beta_1\gamma\mathcal{B}_1(K_D)X^{\beta_1}}{K_D^2(1-\gamma K_D)^3}\left[-\gamma K_D(1+\beta_1)(1-\gamma K_D)+\gamma K_D(2-\gamma K_D-\beta_1\gamma K_D)(2-\beta_1)\right]\\
=&\frac{-\beta_1\gamma^2K_D\mathcal{B}_1(K_D)X^{\beta_1}}{K_D^2(1-\gamma K_D)^3}\left[3(1-\beta_1)+\gamma K_D(\beta_1^2-1)\right]\\
=&\frac{\beta_1\,(\beta_1-1)\,\gamma^2X^{\beta_1}}{K_D(1-\gamma K_D)^3}\left[3-(1+\beta_1)\gamma K_D\right]\mathcal{B}_1(K_D).
\end{aligned}
$$

From the third order partial derivative, it follows that the second order derivative decreases for $K_D<3/(\gamma+\gamma\beta_1)$ and increases for $K_D>3/(\gamma+\gamma\beta_1)$. Next, we try to show that $\frac{\partial^2}{\partial K_D^2}[V(K_D,X)-\delta K_D]$ is negative when $K_D=0$ and $K_D=1/\gamma$. Since

$$\frac{\partial^2\left[V(X,K_D)-\delta K_D\right]}{\partial K_D^2}$$

$$
\begin{aligned}
=&\frac{-\beta_1\gamma(2-\gamma K_D-\beta_1\gamma K_D)X^{\beta_1}}{K_D(1-\gamma K_D)^2}\left[-\frac{K_D(1-\gamma K_D)X_F^{*1-\beta_1}(K_D)}{2(r-\alpha)}\right.\\
&+\frac{cK_D X_F^{*-\beta_1}(K_D)}{2r}+\frac{cK_D X_F^{*\beta_2-\beta_1}(K_D)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{c}{1-\gamma K_D}\right)^{-\beta_2}\\
&\left.-\frac{cK_D}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-\gamma K_D-2\gamma K_F^*(K_D)}\right)^{-\beta_1}\right]-\frac{2\gamma X}{r-\alpha},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left.\frac{\partial^2\left[V(X,K_D)-\delta K_D\right]}{\partial K_D^2}\right|_{K_D=0}\\
=&-2\beta_1\gamma X^{\beta_1}\left[-\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}\right.\\
&\left.+\frac{cX_F^{*\beta_2-\beta_1}(0)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)c^{-\beta_2}-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}\right]\\
&-\frac{2\gamma X}{r-\alpha}\\
=&-2\beta_1\gamma X^{\beta_1}\left[-\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}\right.\\
&+\frac{cX_F^{*\beta_2-\beta_1}(0)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)c^{-\beta_2}-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}\\
&\left.+\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)}\right].
\end{aligned}
$$

Because $-2\beta_1\gamma X^{\beta_1}<0$, in order to show that $\partial^2\left[V(X,K_D)-\delta K_D\right]/\partial K_D^2\big|_{K_D=0}<0$, we need to derive

$$
\begin{aligned}
&-\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}\\
&+\frac{c^{1-\beta_2}X_F^{*\beta_2-\beta_1}(0)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}+\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)}>0.
\end{aligned}
$$

Under the entry deterrence strategy, we have inequality relation

$$
\frac{c}{1-2\gamma K_F^*(0)}>X_F^*(0)>X>c.
$$

Thus,

$$
\begin{aligned}
& -\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1} \\
& +\frac{cX_F^{*\beta_2-\beta_1}(0)}{2(\beta_1-\beta_2)}\underbrace{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)}_{<0}c^{-\beta_2}-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}+\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)} \\
> & -\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1} \\
& +\frac{cX_F^{*\beta_2-\beta_1}(0)}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)X_F^{*-\beta_2}(0)-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}+\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)} \\
> & -\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_2-1}{r-\alpha}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}+\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_2}{r}X_F^{*-\beta_1}(0) \\
& +\frac{c}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)X_F^{*-\beta_1}(0)-\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{cX_F^{*-\beta_1}(0)}{2r}+\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)} \\
= & -\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_2-1}{r-\alpha}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}+\underbrace{\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_1-1}{r-\alpha}}_{>0}X_F^{*-\beta_1}(0) \\
& -\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\underbrace{\frac{X^{1-\beta_1}}{\beta_1(r-\alpha)}}_{>0} \\
> & -\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_2-1}{r-\alpha}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}+\frac{c}{2(\beta_1-\beta_2)}\frac{\beta_1-1}{r-\alpha}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1} \\
& -\frac{X_F^{*1-\beta_1}(0)}{2(r-\alpha)}+\frac{X_F^{*1-\beta_1}(0)}{\beta_1(r-\alpha)} \\
= & \frac{c}{2(r-\alpha)}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}+\frac{X_F^{*1-\beta_1}(0)}{r-\alpha}\left(\frac{1}{\beta_1}-\frac{1}{2}\right).
\end{aligned}
$$

In the following analysis, we show it numerically that

$$
\mathcal{G}=\frac{c}{2}\left(\frac{c}{1-2\gamma K_F^*(0)}\right)^{-\beta_1}+X_F^{*1-\beta_1}(0)\left(\frac{1}{\beta_1}-\frac{1}{2}\right)>0. \tag{3.91}
$$

The default parameter values are $\alpha=0.05$, $r=0.1$, $\sigma=0.2$, $c=2$, $\delta=10$, and $\gamma=0.05$. Note that $\mathcal{G}$ does not depend on γ, this is because from the monopolistic firm's investment decision

$$1-2\gamma K_F^*(0)=\frac{c}{X_F^*(0)}\left(\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right)^{\frac{1}{\beta_1}}.$$

Thus, (3.91) can be written as

$$\mathcal{G}=\frac{\delta(\beta_1-\beta_2)}{(1+\beta_1)F(\beta_2)}X_F^{*-\beta_1}(0)+X_F^{*1-\beta_1}(0)\left(\frac{1}{\beta_1}-\frac{1}{2}\right),$$

with $X_F^*(0)$ being determined by the following implicit expression

$$\frac{c^{1-\beta_2}F(\beta_1)X_F^{*\beta_2}(0)}{\beta_1}+\frac{\beta_1-1}{\beta_1}\frac{X_F^*(0)}{r-\alpha}-\frac{2c}{r}+\frac{\beta_1+1}{\beta_1}\frac{c^2}{X_F^*(0)(r+\alpha-\sigma^2)}-2\delta$$
$$+\frac{2\delta c}{X_F^*(0)}\left(\frac{2\delta(\beta_1-\beta_2)}{c(1+\beta_1)F(\beta_2)}\right)^{\frac{1}{\beta_1}}=0.$$

Figure 3.22 shows that (3.91) holds for the change in the corresponding parameter. So, we can assume that $\partial^2\left[V(X,K_D)-\delta K_D\right]/\partial K_D^2\big|_{K_D=0}<0$.

We then need to show that $\lim_{K_D\to 1/\gamma}\frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2}<0$. First, note that

$$\begin{aligned}
&\lim_{K_D\to 1/\gamma}\frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2}\\
=&\lim_{K_D\to 1/\gamma}\frac{\beta_1\gamma(\beta_1-1)}{(1-\gamma K_D)^2}\frac{c}{2X_F^{*\beta_1}(K_D)}\left[-\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_1}\right.\\
&\left.+\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\frac{1}{r-\alpha}\frac{X_F^*(0)}{X_1(0)}+\frac{1}{r}\right]\\
=&\lim_{K_D\to 1/\gamma}\frac{\beta_1\gamma(\beta_1-1)X_F^{*2-\beta_1}(K_D)}{2c}\left(\frac{X_1(0)}{X_F^*(0)}\right)^2\left[-\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_1}\right.\\
&\left.+\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\frac{1}{r-\alpha}\frac{X_F^*(0)}{X_1(0)}+\frac{1}{r}\right].
\end{aligned}$$

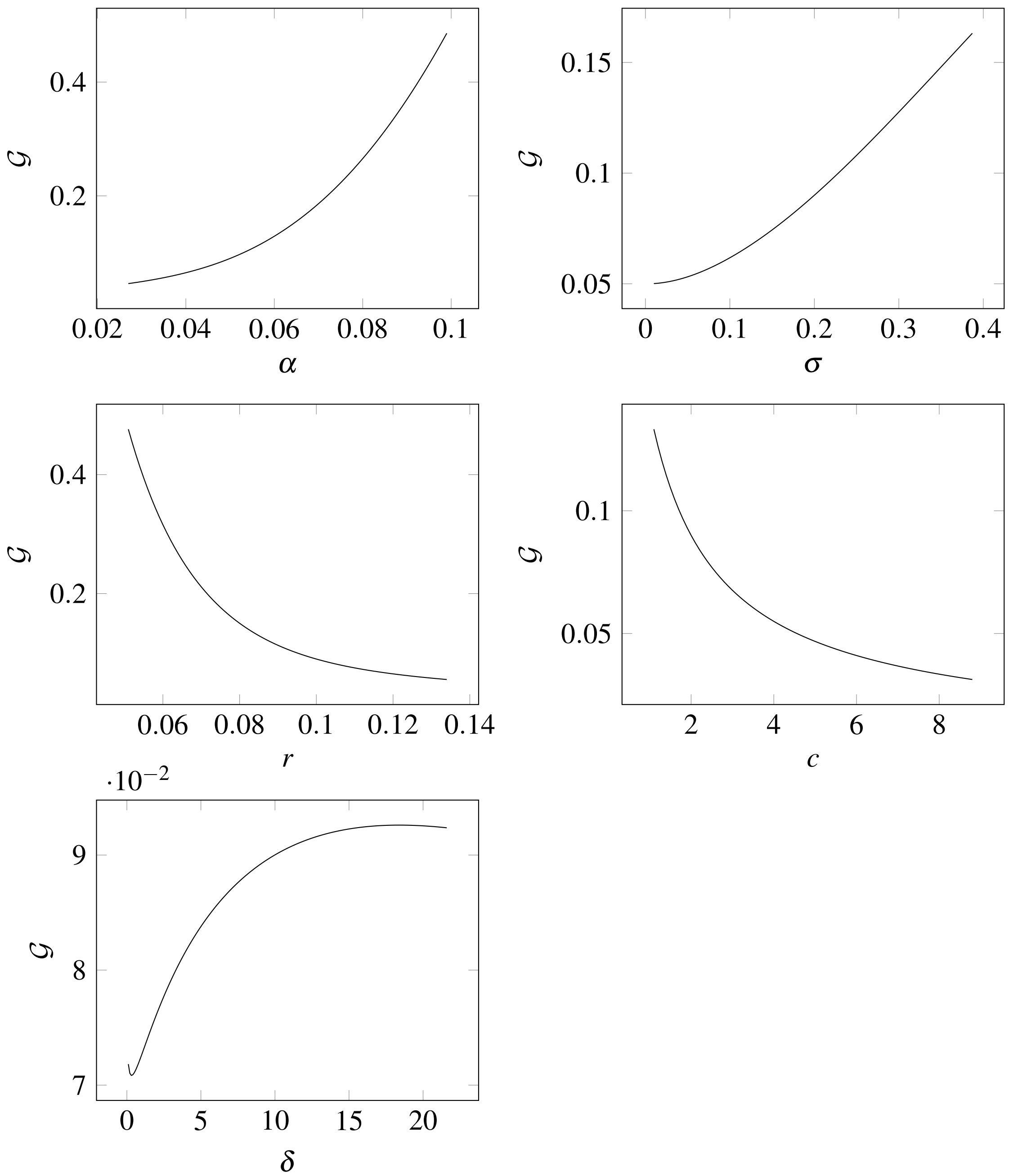


Figure 3.22: Illustration of $\mathcal{G} > 0$. Default parameter values are $\alpha = 0.05$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$.

If $\beta_1 < 2$, then $\lim\limits_{K_D \to 1/\gamma} X_F^{*2-\beta_1}(K_D) \to \infty$. Because

$$\left(\frac{X_1(0)}{X_F^*(0)}\right)^2 \left[-\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_2-1}{r-\alpha}-\frac{\beta_2}{r}\right)\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_1}\right.$$
$$\left. +\frac{1}{\beta_1-\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2} - \frac{1}{r-\alpha}\frac{X_F^*(0)}{X_1(0)} + \frac{1}{r}\right]$$
$$< 0,$$

from $\mathcal{B}_1(K_D) < 0$, it can be concluded that $\lim_{K_D \to 1/\gamma} \frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2} \to -\infty$. If $\beta_1 = 2$, then it still holds that $\lim_{K_D \to 1/\gamma} \frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2} < 0$. If $\beta_1 > 2$, then $X_F^{*2-\beta_1}(K_D) \to 0$ if $K_D \to 1/\gamma$, and $\lim_{K_D \to 1/\gamma} \frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2} = 0$. Thus, we have

$$\lim_{K_D \to 1/\gamma} \frac{\partial^2}{\partial K_D^2}[V(X,K_D) - \delta K_D] \leq 0.$$

Proof of negative second order partial derivative under the entry accommodation strategy Under the entry accommodation strategy when the follower produces below capacity right after investment, the second order partial derivative of $V(X,K_D) - \delta K_D$ with respect to K_D is

$$\begin{aligned}
&\frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2} \\
=&\frac{(-\gamma-\beta_1\gamma)K_D(1-\gamma K_D)-(1-\gamma K_D-\beta_1\gamma K_D)(1-2\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_1(K_D)X^{\beta_1} \\
&+\frac{(1-\gamma K_D-\beta_1\gamma K_D)^2}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_1(K_D)X^{\beta_1}+\frac{(1-\gamma K_D-\beta_2\gamma K_D)^2}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_2(K_D)X^{\beta_2} \\
&+\frac{-\gamma K_D(1+\beta_2)(1-\gamma K_D)-(1-\gamma K_D-\beta_2\gamma K_D)(1-2\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_2(K_D)X^{\beta_2}-\frac{\gamma X}{r-\alpha} \\
=&\frac{-\gamma K_D(1+\beta_1)(1-\gamma K_D)+(1-\gamma K_D-\beta_1\gamma K_D)\gamma K_D(1-\beta_1)}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_1(K_D)X^{\beta_1} \\
&+\frac{-\gamma K_D(1+\beta_2)(1-\gamma K_D)+(1-\gamma K_D-\beta_2\gamma K_D)\gamma K_D(1-\beta_2)}{K_D^2(1-\gamma K_D)^2}\mathcal{M}_2(K_D)X^{\beta_2}-\frac{\gamma X}{r-\alpha} \\
=&\frac{-\beta_1\gamma(2-\gamma K_D-\beta_1\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{M}_1(K_D)X^{\beta_1}+\underbrace{\frac{-\beta_2\gamma(2-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{M}_2(K_D)X^{\beta_2}}_{<0}-\frac{\gamma X}{r-\alpha}.
\end{aligned}$$

If $2-\gamma K_D-\beta_1\gamma K_D \geq 0$, from $\mathcal{M}_1(K_D)>0$, it follows that

$$\frac{\partial^2}{\partial K_D^2}[V(X,K_D)-\delta K_D]<0.$$

If $2-\gamma K_D-\beta_1\gamma K_D<0$, i.e., $\frac{2}{\gamma(1+\beta_1)}<K_D<\frac{1}{\gamma}$, this means that K_D is relative large. Let's see whether the entry accommodation strategy is possible in this case. First notice that, in this case, we have

$$1-\gamma K_D-\beta_1\gamma K_D<0.$$

Since $\mathcal{M}_1(K_D)>0$,

$$\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)X^{\beta_1}<0.$$

From $\mathcal{M}_2(K_D)<0$, and $\beta_2<0$, it holds that

$$\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)X^{\beta_2}<0.$$

Also $K_D>\frac{2}{(1+\beta_1)\gamma}$ means that

$$\frac{1-2\gamma K_D}{2(r-\alpha)}<\frac{\beta_1-3}{2(\beta_1+1)(r-\alpha)}.$$

Next, we show that $\beta_1\leq 3$, so that $\frac{1-2\gamma K_D}{2(r-\alpha)}<0$. In the next, numerical analysis will be carried out to show that $\beta_1<3$. The default parameter values we use are $\alpha=0.05$, $r=0.1$, $\sigma=0.2$, $c=2$, $\delta=10$, and $\gamma=0.05$. Since β_1 is only changing with r, α, and σ, Figure 3.23 shows β_1 as functions of r, α, and σ. The figure shows that β_1 is smaller than 3.

Overall, the above three negative terms imply that

$$\frac{1-\gamma K_D-\beta_1\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_1(K_D)X^{\beta_1}+\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{M}_2(K_D)X^{\beta_2}$$
$$+\frac{1-2\gamma K_D}{2(r-\alpha)}-\frac{c}{2r}-\delta<0.$$

So, this subcase does not give admissible entry accommodation capacity.

Overall, if there is an admissible solution such that it is not bigger than $\frac{2}{(1+\beta_1)\gamma}$, then it would maximise $V_D(X,K_D)-\delta K_D$.

Producing up to capacity right after investment

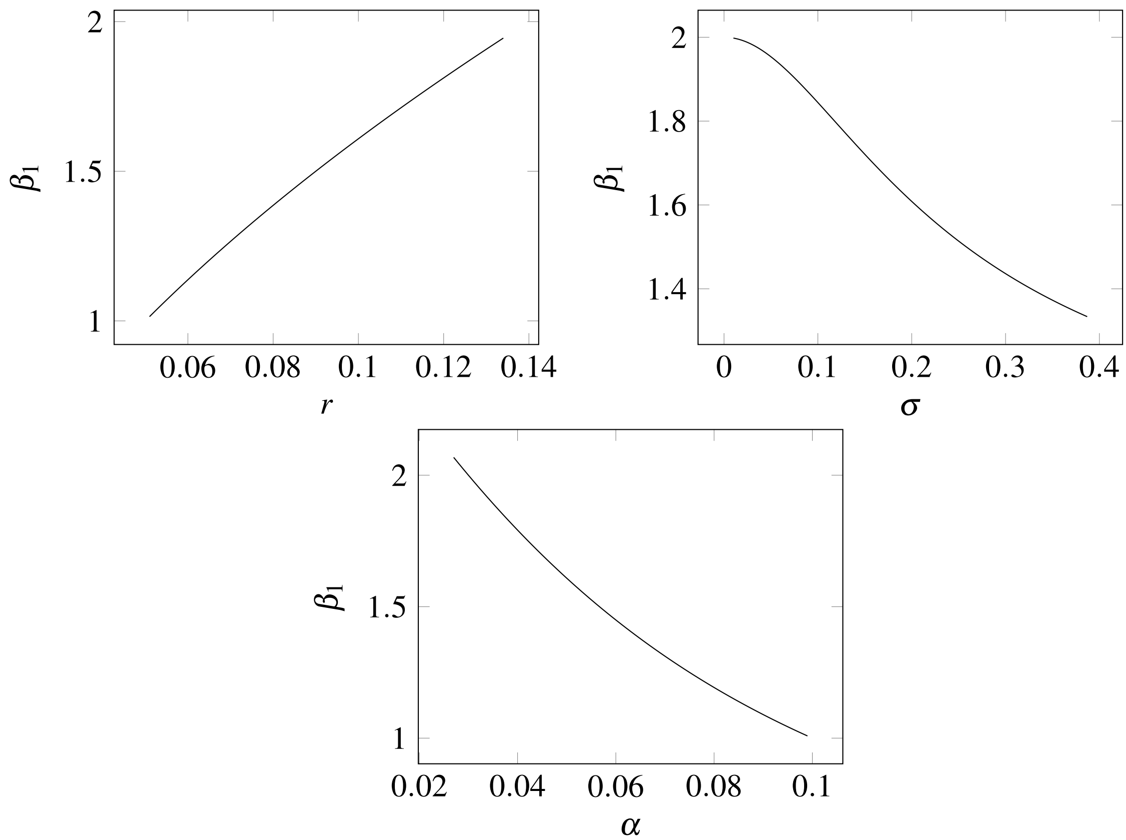


Figure 3.23: Illustration of β_1 smaller than 3. Default parameter values are $\alpha = 0.05$, $r = 0.1$, $\sigma = 0.2$.

Proof of negative second order partial derivative under the entry deterrence strategy
Under the entry deterrence strategy when the follower produces up to capacity right after investment, the second order derivative of $V_D(X,K_D) - \delta K_D$ with respect to K_D is

$$\begin{aligned}
&\frac{\partial^2[V_D(X,K_D)-\delta K_D]}{\partial K_D^2}\\
=&\frac{-(1-\gamma K_D)+\gamma K_D(1-\gamma K_D-\beta_1\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{B}_2(K_D)X^{\beta_1}\\
&+\frac{(1-\gamma K_D-\beta_1\gamma K_D)^2}{K_D^2(1-\gamma K_D)^2}\mathcal{B}_2(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
=&-\frac{\beta_1\gamma(2-\gamma K_D-\beta_1\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{B}_2(K_D)X^{\beta_1}-\frac{2\gamma X}{r-\alpha}.
\end{aligned}$$

The third order partial derivative with respect to K_D is

$$\frac{\partial^3[V_D(X,K_D)-\delta K_D]}{\partial K_D^3}=\frac{\beta_1(\beta_1-1)\gamma^2X^{\beta_1}}{K_D(1-\gamma K_D)^3}\left(3-(1+\beta_1)\gamma K_D\right)\mathcal{B}_1(K_D).$$

Because $\mathcal{B}_2(K_D)$ is negative, the second order derivative decreases for $0 < K_D < \frac{3}{(1+\beta_1)\gamma}$ and increases for $\frac{3}{(1+\beta_1)\gamma} < K_D < \frac{1}{\gamma}$. Similar to the proof of negative second order derivative for the deterrence strategy when the flexible follower produces below capacity right after investment, we want to show that

$$\begin{aligned}
\left.\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2}\right|_{K_D=0}&<0,\\
\lim_{K_D\to 1/\gamma}\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2}&<0.
\end{aligned}$$

We first show that $\left.\frac{\partial^2\ V_D(K_D,X)-\delta K_D}{\partial K_D^2}\right|_{K_D=0} < 0$. First note that in this case, we have

$$X_1(0)=c<X_2(0)=\frac{c}{1-2\gamma K_F^*(0)}\leq X<X_F^*(0).$$

From

$$X^{det}(K_D)=\frac{\beta_1(r-\alpha)}{(\beta_1-1)(1-\gamma K_D)}\left(\frac{c}{r}+\delta\right),$$

it follows that

$$X \geq \frac{\beta_1(r-\alpha)}{(\beta_1-1)}\left(\frac{c}{r}+\delta\right).$$

The second order derivative when $K_D = 0$ is

$$\begin{aligned}
&\left.\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2}\right|_{K_D=0}\\
=&-\frac{\gamma\beta_1 c}{\beta_1-\beta_2}\left\{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_2}\right)\right.\\
&\left.+\frac{\beta_1-\beta_2}{r-\alpha}\left(\frac{X_F^*(0)}{X_2(0)}-\frac{X_F^*(0)}{X_1(0)}\right)\right\}\left(\frac{X}{X_F^*(0)}\right)^{\beta_1}-\frac{2\gamma X}{r-\alpha}\\
<&-\frac{\gamma\beta_1 c}{\beta_1-\beta_2}\left\{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_2}\right)\right.\\
&\left.+\frac{\beta_1-\beta_2}{r-\alpha}\left(\frac{X_F^*(0)}{X_2(0)}-\frac{X_F^*(0)}{X_1(0)}\right)\right\}-\frac{2\beta_1}{(\beta_1-1)}\left(\frac{c}{r}+\delta\right)=\mathcal{G}.
\end{aligned}$$

This means that if we can show $\mathcal{G} \leq 0$, then it would imply that $\left.\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2}\right|_{K_D=0} < 0$. In Figure 3.24, we show numerically that $\mathcal{G}$ is not positive over α, σ, r, c, δ, and γ. The default parameter values are $\alpha = 0.02$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$, and $\gamma = 0.05$.

We also need to show $\lim_{K_D\to 1/\gamma}\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2} < 0$. Because $\frac{X_F^*(K_D)}{X_1(K_D)}$ and $\frac{X_F^*(K_D)}{X_2(K_D)}$ are constants, the second order derivative of $V_D(K_D,X)-\delta K_D$ as $K_D \to 1/\gamma$ can be written as

$$\begin{aligned}
&\lim_{K_D\to 1/\gamma}\frac{\partial^2\, V_D(K_D,X)-\delta K_D}{\partial K_D^2}\\
=&\lim_{K_D\to 1/\gamma}\frac{\gamma\beta_1(\beta_1-1)X^{\beta_1}}{2c(\beta_1-\beta_2)X_F^{*\beta_1-2}(K_D)}\left(\frac{X_1(0)}{X_F^*(0)}\right)^2\left\{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_2}\right)\right.\\
&\left.+\frac{\beta_1-\beta_2}{r-\alpha}\left(\frac{X_F^*(0)}{X_2(0)}-\frac{X_F^*(0)}{X_1(0)}\right)\right\}-\frac{2\gamma X}{r-\alpha}.
\end{aligned}$$

Since $\mathcal{B}_2(K_D) < 0$, we have

$$\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left(\left(\frac{X_F^*(0)}{X_1(0)}\right)^{\beta_2}-\left(\frac{X_F^*(0)}{X_2(0)}\right)^{\beta_2}\right)+\frac{\beta_1-\beta_2}{r-\alpha}\left(\frac{X_F^*(0)}{X_2(0)}-\frac{X_F^*(0)}{X_1(0)}\right) < 0.$$

Thus, it holds that $\lim_{K_D\to 1/\gamma}\frac{\partial^2[V_D(K_D,X)-\delta K_D]}{\partial K_D^2} < 0$.

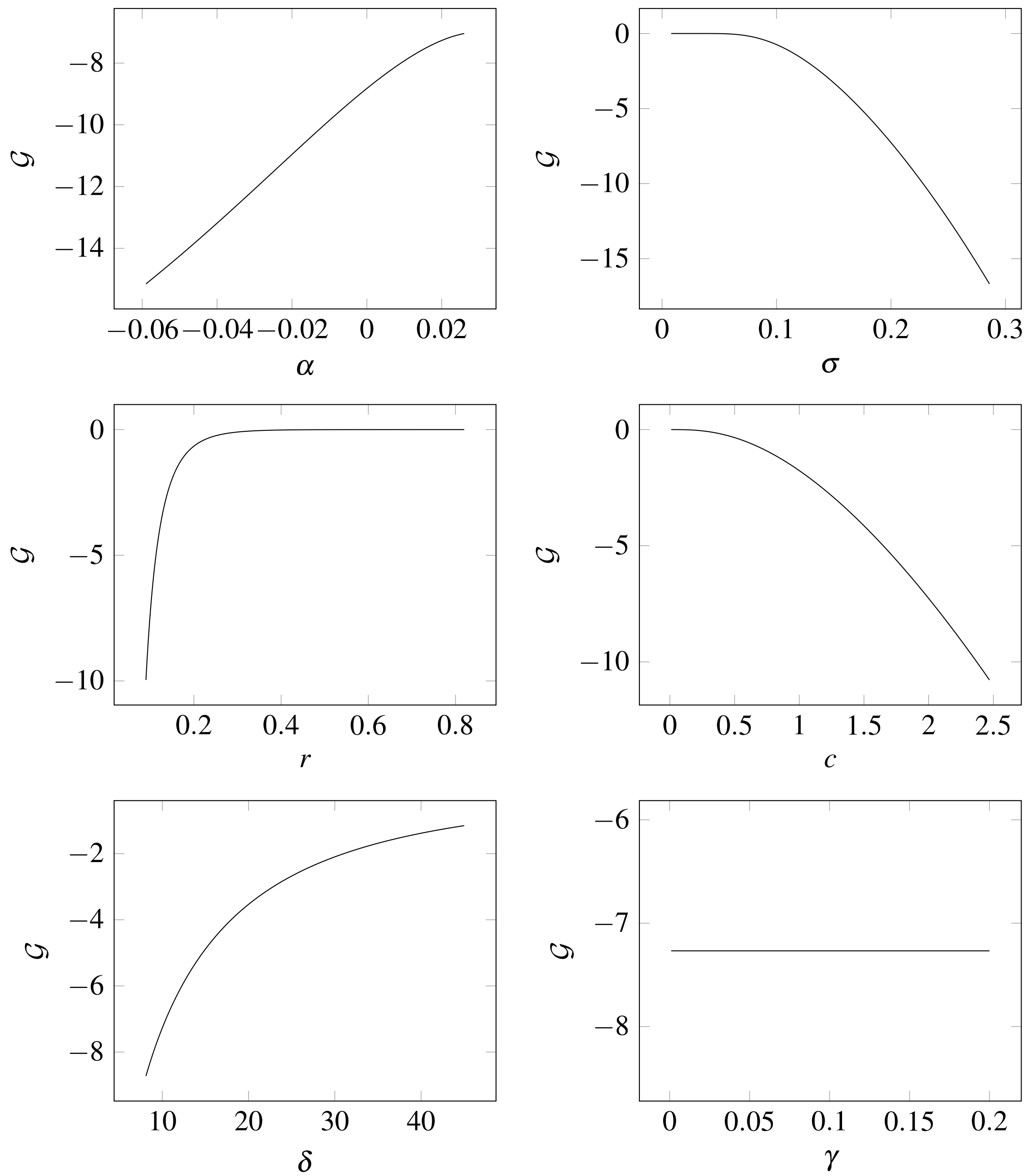


Figure 3.24: Illustration of negative $\mathcal{G}$. Default parameter values are $\alpha = 0.02$, $r = 0.1$, $\sigma = 0.2$, $c = 2$, $\delta = 10$, $\gamma = 0.05$.

Proof of negative second order partial derivative under the entry accommodation strategy Under the entry accommodation strategy when the follower produces up to capacity right after investment, the second order derivative of $\mathcal{N}(K_D)$ with respect to K_D can be derived that

$$\begin{aligned}\frac{d^2\mathcal{N}(K_D)}{dK_D^2} =& \frac{(-\gamma-\beta_2\gamma)K_D(1-\gamma K_D)-(1-\gamma K_D-\beta_2\gamma K_D)(1-2\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{N}(K_D)\\ &+\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\frac{1-\gamma K_D-\beta_2\gamma K_D}{K_D(1-\gamma K_D)}\mathcal{N}(K_D)\\ =&\frac{(1-\gamma K_D-\beta_2\gamma K_D)(\gamma K_D-\beta_2\gamma K_D)-(1-\gamma K_D)(\gamma K_D+\beta_2\gamma K_D)}{K_D^2(1-\gamma K_D)^2}\mathcal{N}(K_D)\\ =&\frac{-\gamma\beta_2(2-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{N}(K_D).\end{aligned}$$

Besides, it holds that

$$\begin{aligned}&\frac{d}{dK_D}\frac{(1-2\gamma K_D)(1-\gamma K_D-\gamma K_F^*(K_D))}{1-\gamma K_D}\\ =&\frac{1}{(1-\gamma K_D)^2}\Big\{\left[-2\gamma-\gamma(1-2\gamma K_D)\frac{1}{1-\gamma K_D}\right](1-\gamma K_D)(1-\gamma K_D-\gamma K_F^*(K_D))\\ &+\gamma(1-2\gamma K_D)(1-\gamma K_D-\gamma K_F^*(K_D))\Big\}\\ =&\frac{1-\gamma K_D-\gamma K_F^*(K_D)}{(1-\gamma K_D)^2}[-\gamma(3-4\gamma K_D)+\gamma(1-2\gamma K_D)]\\ =&\frac{\gamma(1-\gamma K_D-\gamma K_F^*(K_D))(-2+2\gamma K_D)}{(1-\gamma K_D)^2}=\frac{-2\gamma(1-\gamma K_D-\gamma K_F^*(K_D))}{1-\gamma K_D}.\end{aligned}$$

The second order partial derivative of $V_D(X,K_D)-\delta K_D$ with respect to K_D is

$$\begin{aligned}&\frac{\partial^2 V_D(X,K_D)-\delta K_D}{\partial K_D^2}\\ =&\frac{d^2\mathcal{N}(K_D)}{dK_D^2}X^{\beta_2}-\frac{2\gamma X}{r-\alpha}\frac{1-\gamma K_D-\gamma K_F^*(K_D)}{1-\gamma K_D}\\ =&\frac{-\gamma\beta_2(2-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)^2}\mathcal{N}(K_D)X^{\beta_2}-\frac{2\gamma X}{r-\alpha}\frac{1-\gamma K_D-\gamma K_F^*(K_D)}{1-\gamma K_D}\end{aligned}$$

From $\beta_2<0$ and $\mathcal{N}(K_D)>0$, then it follows that

$$\frac{\partial^3\mathcal{N}(K_D)}{\partial K_D^3}=-\gamma\beta_2\frac{-(\gamma+\gamma\beta_2)K_D(1-\gamma K_D)-(2-\gamma K_D-\beta_2\gamma K_D)(1-3\gamma K_D)}{K_D^2(1-\gamma K_D)^3}\mathcal{N}(K_D)$$

$$
\begin{aligned}
&+\frac{-\gamma\beta_2(2-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)^2}\frac{(1-\gamma K_D-\beta_2\gamma K_D)}{K_D(1-\gamma K_D)}\mathcal{N}(K_D)\\
=&\frac{-\gamma\beta_2\mathcal{N}(K_D)\gamma K_D}{K_D^2(1-\gamma K_D)^3}[(2-\gamma K_D-\beta_2\gamma K_D)(2-\beta_2)-(1-\gamma K_D)(1+\beta_2)]\\
=&\frac{-\gamma^2\beta_2\mathcal{N}(K_D)(1-\beta_2)[3-(1+\beta_2)\gamma K_D]}{K_D(1-\gamma K_D)^3}>0
\end{aligned}
$$

and

$$
\frac{\partial^3[V_D(X,K_D)-\delta K_D]}{\partial K_D^3}=\frac{-\gamma^2\beta_2(1-\beta_2)[3-(1+\beta_2)\gamma K_D]}{K_D(1-\gamma K_D)^3}X^{\beta_2}\mathcal{N}(K_D)>0.
$$

The positive third order derivative implies that the second order derivative of $V_D(X,K_D)-\delta K_D$ increases for all $K_D\in[0,1/\gamma)$. If it can be shown that the second order partial derivative is negative when K_D approaches $1/\gamma$, then it holds on the interval of $[0,1/\gamma)$ that the second order partial derivative is negative. The second order partial derivative when $K_D\to 1/\gamma$ is

$$
\begin{aligned}
&\lim_{K_D\to 1/\gamma}\frac{\partial^2[V_D(X,K_D)-\delta K_D]}{\partial K_D^2}\\
=&\lim_{K_D\to 1/\gamma}\frac{-\beta_2\gamma(2-\gamma K_D-\beta_2\gamma K_D)}{(1-\gamma K_D)^2}\frac{c^{1-\beta_2}X^{\beta_2}}{2(\beta_1-\beta_2)}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\Big[(1-\gamma K_D)^{\beta_2}\\
&\quad-(1-\gamma K_D-2\gamma K_F^*(K_D))^{\beta_2}\Big]-\frac{2\gamma X}{r-\alpha}\frac{1-\gamma K_D-\gamma K_F^*(K_D)}{1-\gamma K_D}\\
=&\lim_{K_D\to 1/\gamma}\frac{-c\beta_2\gamma(1-\beta_2)X^{\beta_2}}{2(\beta_1-\beta_2)(1-\gamma K_D)^2}\left(\frac{X}{X_1(K_D)}\right)^{\beta_2}\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[1-\left(\frac{X_1(K_D)}{X_2(K_D)}\right)^{\beta_2}\right]\\
&\quad-\frac{2\gamma X}{r-\alpha}\frac{1-\gamma K_D-\gamma K_F^*(K_D)}{1-\gamma K_D}\\
<&\lim_{K_D\to 1/\gamma}\underbrace{\frac{-c\beta_2\gamma(1-\beta_2)X^{\beta_2}}{2(\beta_1-\beta_2)(1-\gamma K_D)^2}\left(\frac{X}{X_1(K_D)}\right)^{\beta_2}}_{>0}\underbrace{\left(\frac{\beta_1-1}{r-\alpha}-\frac{\beta_1}{r}\right)\left[1-\left(\frac{X_1(K_D)}{X_2(K_D)}\right)^{\beta_2}\right]}_{<0}\\
&\quad-\underbrace{\frac{2\gamma X}{r-\alpha}\frac{X_1(K_D)}{X_2(K_D)}}_{>0}.
\end{aligned}
$$

Thus,

$$
\lim_{K_D\to 1/\gamma}\frac{\partial^2[V(X,K_D)-\delta K_D]}{\partial K_D^2}<0.
$$

In conclusion, it follows that for all $0 \leq K_D < 1/\gamma$,

$$\frac{\partial^2 [V(X, K_D) - \delta K_D]}{\partial K_D^2} < 0.$$

CHAPTER 4

Subsidized Capacity Investment under Uncertainty

This chapter studies how the subsidy support, e.g. price support and reimbursed investment cost support, affects the investment decision of a monopoly firm under uncertainty and analyzes the implications for social welfare. The analytical results show that the unconditional, i.e., subsidy support that is introduced from the beginning, makes the firm invest earlier. Under a linear demand structure, the unconditional subsidy cannot align the firm's investment decision to the social optimal one. However, a conditional subsidy, i.e., subsidy support introduced at the social optimal investment threshold, can align the two decisions. For a non-linear demand structure, it is possible for the unconditional subsidy to make the firm invest according to the social optimum. When the investment decisions are aligned, the firm's investment leads to the first-best outcome.

4.1 Introduction

Since the 1970s, many public owned functions and businesses have been decentralized, e.g. postal services, banking, airlines, telecommunication, and public infrastructures. The government owns many resources such as water, land, and mineral. Decentralization means the private firms have the right to invest, produce, and make profit from such resources, for example, port investment, agriculture investment, green energy investment, and so on. The privatization is believed to be more efficient and effective in decision making because of the quicker reaction to unanticipated market changes. However, after the privatization, firms prioritize profit maximization and do not consider the social optimal goals when making decisions. This is different from the goal of the social planner, which is to achieve social optimality. For instance, energy producers that use fossil fuel and emit greenhouse gases do not take into account environmental damage (Eichner and Runkel, 2014), whereas the regulator such as the E.U. parliament, has the purpose to fulfill the emissions reduction commitment and encourage the investments of renewable energy. In risky environments it has been argued that a firm tends to postpone investment (see e.g., McDonald and Siegel (1986)). According toDobbs (2004), the level of investment in capacity might also be constrained by the firm. For example, an electricity producer might hesitate to invest in renewable technology due to high investment costs compared to the fossil fuels. This implies that the energy market has less incentive to deliver the desired level of renewable investment. This difference in objectives and investment strategies between the profit and welfare maximizers poses a coordination problem and requires governmental regulation (Rodrik, 1992).

In a market with uncertain future demand, the firm is constantly forecasting demand and balancing the value of investing now and delaying investments. Thus, the real options approach is used to analyze the investment decisions. Several literatures have studied to use price regulation such as the price cap to regulate the delayed investment under uncertainty. For instance, Dobbs (2004) argues that the first-best outcome cannot be reached as price cap is used for two goals: optimal investment ex-ante and optimal post-investment pricing. Building on Dobbs (2004), Evans and Guthrie (2012) show that the price cap should be lowered under scale economics where grouping investments across time is cost efficient. By contrast, Willems and Zwart (2017) consider constant returns to scale where it is not optimal to group investments. By assuming asymmetric information on investment costs, Willems and Zwart (2017) study the optimal mechanism where a revenue tax increases with the level of the price cap. In this chapter, we study the policy instrument of subsidy, rather than price cap.

Subsidy support is a very common policy instrument in the fields of agriculture and green energy. For agriculture in developing economies, there are input subsidies, which are implemented as price subsidies accessible to producers according to Chirwa and Dorward (2013). One example is the Indian fertilizer subsidy in order to encourage the domestic production of fertilizer and to increase its use. To accomplish these two objectives, India introduced the RPS (Retention Price Subsidy) scheme in 1977, where the difference between retail price and retention price (adjusted for freight and dealer's margin) was paid back to the manufactures as a subsidy (Sharma and Thaker, 2010). Under the RPS, the production cost plus 12% profit is covered by the subsidy. Later on, RPS was criticized for being inefficient to motivate the producers to decrease production costs and was replaced by NPS (New Pricing Scheme) in 2003. Under the new system, the producer receives a set amount based on the age of the production plant and the amount of feedstock used.

In the green energy field, the subsidy support can take many forms such as feed-in premiums, reimbursed investment costs, feed-in tariffs, tradable green certificates, and quota obligations. In this research work, two kinds of price support will be discussed: flexible and fixed price support. Under flexible price support, the producer receives a payment proportional to the market price for every product sold to the consumer, like 12% for instance in India's RPS scheme. Under the fixed price support, the producer receives a fixed payment for every product sold to the consumer that is independent of the market price, like the subsidy described in India's NPS scheme. In the green energy field, the fixed price support may take the form as the feed-in premium subsidy.

This research studies how different kinds of subsidy support affect the profit maximizing firm's investment timing and size, and whether it is possible to align the firm's investment decisions to the social optimal ones. Besides the non-linear demand structure used in the price cap literatures, this chapter considers also the linear demand structure. More specifically, we consider two kinds of demand shocks for the linear demand: additive (Kulatilaka and Perotti, 1998; Aguerrevere, 2003; Hagspiel et al., 2016) and multiplicative (Grenadier, 2000; Huisman and Kort, 2015) demand shocks. We show that the subsidy policies introduced from the beginning make the firm invest earlier and invest less. Moreover, we find that there exists a conditional subsidy to introduce the subsidy support at the social optimal investment timing to align the firm's optimal investment decision to the social optimal one. For the non-linear demand structure, if the demand is iso-elastic as in Aguerrevere (2009) and Novy-Marx (2007), the influence of the subsidy on the firm's investment decision depends on the subsidy rate. It is possible to align the firm's investment decision to the social optimal decision if subsidies are introduced from the beginning, or at the social optimal investment timing. For both demand structures, the subsidies that align firm's and social optimal investment decision yield the social optimal surplus. To simplify the analysis, we do not consider the efficiency loss in collecting taxes and the allocation of taxation as subsidies.

Several research papers have already shed light on investment decisions under policy schemes in the framework of real options. For the policy scheme that will prevail once being chosen, Pennings (2000) studies the taxation and investment subsidy to stimulate the instant investment, i.e., the waiting time is zero. Hassett and Metcalf (1999) consider the uncertainty in the tax policy, such as the U.S. investment tax credits that have been changed on many occasions since being introduce in 1964, and show that for a relatively low tax rate, more uncertainty in tax policy speeds up irreversible investment because the firm inclines to invest at a low tax rate. This chapter focuses on the subsidized investment and the corresponding welfare analysis, rather than on the taxation.

Most of the existing research concerning policy schemes focuses on green investment and takes the subsidy payments as a volatile process. Up to our best knowledge, those papers only study the investment decisions from the perspective of the producer and considers mainly the investment timing. For example, Boomsma et al. (2012) assume that the geometric Brownian motion governs the capital cost, electricity prices, and subsidy payments. The support schemes considered include feed-in tariff, flexible price premium, and renewable energy certificates. The three support schemes differ at how much risk the firm is exposed to the market. This is different from our research, where the price volatility is the only risk in the market. Besides the investment timing, we also consider the influence of subsidies on the firm's optimal investment capacity. Moreover, we study the optimal subsidy schemes to make the firm invest in a social optimal way.

Current literatures on subsidy mainly consider the uncertainty about introduction or retraction of subsidy schemes. Boomsma and Linnerud (2015) also examine how the market risk and the policy risk of retractable support schemes affect the investment timing. They find that the risk of subsidy termination speeds up the investment. This result is also supported by Adkins and Paxson (2015). They provide the intuition that the firm wants to catch the subsidy before it is gone. Similarly, future provision of permanent subsidy delays investment because the firm wants to wait for the subsidy. This influence of subsidy retraction and provision is further studied by Chronopoulos et al. (2016). Besides the investment timing, they also consider the influence of policy uncertainty on the investment capacity/size. They find that the future subsidy retraction lowers the amount of installed capacity, and the future subsidy provision raises the incentive to install a larger capacity. In this research, we also consider both investment timing and capacity. Rather than the policy uncertainty, the focus is on the welfare analysis of the investment subsidy and the optimal subsidy policies to align the firm's investment decision to the social optimal investment decision.

This chapter is organized as follows. Section 4.2 describes the profit and welfare maximizers' investment problems and the subsidy support. In Section 4.3 we derive the optimal subsidy policy to align the firm's and social optimal investment decisions, and compare the optimal subsidy support schemes. Section 4.4 studies the optimal subsidy policies and compares them for different demand structures. Section 4.5 concludes.

4.2 Model Setup

Consider a continuous-time and one-time irreversible capacity investment problem. Investor needs to decide on the investment timing and investment capacity. There is no depreciation of capacity and no production costs and the marginal cost of investment is constant, $\delta > 0$. Once a capacity K is installed, K will be sold in the market at a price $p(X(t),K)$. $\{X(t)|t \geq 0\}$ is the demand shift parameter and satisfies a geometric Brownian motion,

$$dX(t) = \mu X(t)\,dt + \sigma X(t)\,d\omega(t), \tag{4.1}$$

in which μ is the drift parameter, $d\omega(t)$ is the increment of a Wiener process, and $\sigma > 0$ is the volatility parameter. The discount rate is r and we assume $r > \mu$. The instant producer surplus is profit flow $p(X(t),K)K$. The instant consumer surplus is denoted as $cs(X(t),K)$. A regulator's objective is to maximize the producer and consumer surplus minus investment costs, i.e.,

$$\max_{T\geq 0, K\geq 0} E\left[\int_{t=T}^{\infty} (p(X(t),K)K + cs(X(t),K))\exp(-rt)\mathrm{d}t - \delta K\exp(-rT)\,\middle|\, X(0)=X\right]. \tag{4.2}$$

This yields the social optimal investment decision (X_W^*, K_W^*) with X_W^* the social optimal investment threshold that triggers investing K_W^* once it is reached. Hence, the social optimal investment time T is the first time that the stochastic process, which starts at $X(0)$ at time zero, reaches X_W^*. The profit maximizer, the firm, has the objective to maximize the producer surplus minus investment costs, i.e.,

$$\max_{T\geq 0, K\geq 0} E\left[\int_{t=T}^{\infty} p(X(t),K)K\exp(-rt)\mathrm{d}t - \delta K\exp(-rT)\,\middle|\, X(0)=X\right]. \tag{4.3}$$

The solution gives firm's optimal investment decision as (X^*, K^*). X^* is the optimal investment threshold and triggers the firm to invest K^* once being reached. Comparing (4.2) and (4.3), it can be concluded that firm's investment decision is not totally aligned with social optimal investment decision such that $X^* = X_W^*$ and $K^* = K_W^*$ both hold. This distortion implies that firm's optimal investment decision generates externality and does not lead to first-best outcome. In order to align these two investment decisions, the regulator needs to make the firm internalize this externality when deciding on investment. Because the difference between the two objectives, (4.2) and (4.3), is consumer surplus, a possible regulation is to propose a contract that specifies a monetary transfer, e.g., a subsidy, to remunerate the firm. Such subsidy scheme can be a subsidy flow $s(X(t),K)$ that satisfies $s(X(t),K) = cs(X(t),K)$, or a lump sum subsidy transfer $s(X,K)$ to the firm when investing at level X with capacity K. Let $S(X,K)$ and $CS(X,K)$ be the discounted expected subsidy and consumer surplus. For both subsidy flow and lump sum subsidy, when firm's investment decision is aligned to the social optimal decision, the following conditions hold,

$$\frac{S(X_W^*, K_W^*)}{\partial X} = \frac{\partial CS(X_W^*, K_W^*)}{\partial X}, \tag{4.4}$$

$$\frac{S(X_W^*, K_W^*)}{\partial K} = \frac{\partial CS(X_W^*, K_W^*)}{\partial K}. \tag{4.5}$$

(4.4) and (4.5) are straightforward outcomes from the maximization of social surplus and producer surplus when the firm internalizes the subsidy. After subsidy, the producer surplus from profit flow is equal to the producer surplus in (4.2). The producer surplus from subsidy is equal to subsidy costs. Consumer surplus after subsidy has the same value as in (4.2). In this way, social surplus reaches the first-best level after subsidy.

Denote a subsidy flow as $s(X(t),K,\tilde{s})$ for given capacity level K and subsidy rate parameter $\tilde{s} \geq 0$. This flow can be implemented in many forms. It can be a flexible price support (a proportional add-on to the market price), i.e., $s(X(t),K,\tilde{s}) = \tilde{s}P(X(t),K)K$, or a fixed price support (a fixed add-on to the market price), i.e., $s(X(t),K,\tilde{s}) = \tilde{s}K$. These two subsidy flows influence firm's profit flow directly because of their relation to market prices. Denote a lump sum subsidy as $s(K,\tilde{s})$. It can be reimbursed investment cost (a one time remuneration transfer as a fraction of investment costs), i.e., $s(K,\tilde{s}) = \tilde{s}\delta K$. Let the expected discounted producer surplus be $V(X,K,\tilde{s})$ for the given geometric Brownian motion level $X(0) = X$ and investment capacity K. The firm's optimal investment decision is $(X^*(\tilde{s}),K^*(\tilde{s}))$ after subsidy $\tilde{s}$. The corresponding expected social surplus is $W(\tilde{s}) = W(X^*(\tilde{s}),K^*(\tilde{s}),\tilde{s})$. When $\tilde{s}^*$ maximizes $W(\tilde{s})$ and yields the firm's optimal decision such that$(X^*(\tilde{s}^*),K^*(\tilde{s}^*)) = (X^*_W,K^*_W)$, then subsidy scheme $\tilde{s}^*$ is optimal. In the following analysis, we focus on the feasibility of these implementations for some specific demand structures.

4.3 Linear Demand

Let the inverse linear demand function for a given investment capacity $K \geq 0$ be

$$\begin{aligned} p(t) =& \alpha[X(t) - \eta K] + (1-\alpha)[X(t)(1-\eta K)] \\ =& X(t) - \eta K(\alpha + (1-\alpha)X(t)),\ 0 \leq \alpha \leq 1,\ \eta > 0 \end{aligned}$$

This demand function combines two types of demand shocks: additive demand shocks $X(t) - \eta K$ and multiplicative demand shocks $X(t)(1-\eta K)$. The additive demand shocks have a weight of α. Besides $r > \mu$, it is assumed that $r > 2\mu + \sigma^2$ holds as in Chapters 2 and 3. For additive demand structure, the market size increases when firm waits for a higher demand level to invest. The additive demand structure corresponds to markets where there is no obvious cap on market size. The multiplicative demand structure is restricted by market size, and it corresponds to a market where the amount of potential customers is limited. An example for multiplicative demand structure is the market of agricultural machines, see Boonman (2014), where the amount of acres of farmlands and the number of farmers are limited. This results in an upper bound of demand.

For the given linear demand function, this section first explores the first-best outcome, where the social planner decides about when and how much to invest. This provides a benchmark for the policy regulator to regulate the monopoly firm. Then the firm's optimal investment decision $(X^*(\tilde{s}),K^*(\tilde{s}))$ is analyzed under monetary subsidy. The analysis focuses specifically on the influence of subsidy on the firm's investment decision, which provides insights on the efficiency of subsidy regulation. Moreover, this section discusses the best performance for unconditional subsidy that is implemented from the beginning and for conditional subsidy that is implemented at some specific demand level.

4.3.1 First-best benchmark

The social planner's maximization problem is described by (4.2). To get a more specific objective function, we first calculate the discounted consumer and producer surplus separately. For the given level of $X(t)$, the instantaneous consumer surplus is

$$cs(X(t),K) = \int_{X(t)-[\alpha+(1-\alpha)X(t)]\eta K}^{X(t)} \frac{X(t)-p}{\alpha\eta+(1-\alpha)X(t)\eta}\mathrm{d}p = \frac{[\alpha+(1-\alpha)X(t)]\eta K^2}{2}.$$

Given $X(0)=X$, the expected discounted consumer surplus is equal to

$$CS(X,K) = E\left[\int_{t=0}^{\infty} \frac{[\alpha+(1-\alpha)X(t)]\eta K^2}{2}\mathrm{d}t \,\middle|\, X(0)=X\right] = \frac{\eta K^2}{2}\left(\frac{\alpha}{r}+\frac{(1-\alpha)X}{r-\mu}\right).$$

For a given X, the consumer surplus increases with investment capacity K. This is because more capacity yields a lower market price since the firm always produces up to full capacity. For a given amount of investment capacity K, the consumer surplus increases with X. The reason is that a higher level X implies a larger market demand. The highest price that consumers are willing to pay increases. The expected producer surplus is equal to the value of the firm, which is the discounted profit flow minus the investment cost. For a given K, the expected producer surplus is

$$PS(X,K) = \frac{XK}{r-\mu} - \frac{(1-\alpha)X\eta K^2}{r-\mu} - \frac{\alpha\eta K^2}{r} - \delta K.$$

The producer surplus increases with X for given K because a larger demand implies a higher market price level, which increases firm's profit flows. The expected social surplus given at $X(0)=X$ is given by

$$W(X,K) = \frac{XK}{r-\mu} - \frac{(1-\alpha)X\eta K^2}{2(r-\mu)} - \frac{\alpha\eta K^2}{2r} - \delta K.$$

From the discounted social welfare function, we can derive the social optimal investment decision as being summarized in the following proposition. The proof can be found in Appendix.

Proposition 4.1. *The social optimal investment threshold X_W^* and the social optimal investment capacity K_W^* satisfy the equations*

$$\frac{X}{r-\mu}\frac{\beta-1}{\beta}(2-(1-\alpha)\eta K)-\frac{\alpha\eta K}{r}-2\delta=0 \tag{4.6}$$

and

$$\alpha(1-\alpha)\eta^2K^2+r\delta(\beta+1)(1-\alpha)\eta K+\alpha(\beta-2)\eta K-2r\delta=0, \tag{4.7}$$

in which

$$\beta=\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}>2.$$

Next we carry out some further analysis on the social optimal investment decision (X_W^*,K_W^*). First, according to Dixit and Pindyck (1994) it holds that $\frac{\partial\beta}{\partial\sigma}<0$ and $\frac{\partial\beta}{\partial\mu}<0$. From equations (4.6) and (4.7), it can be derived that $\frac{\partial X_W^*}{\partial\beta}<0$ and $\frac{\partial K_W^*}{\partial\beta}<0$. Thus, like the standard real options result for firm's investment decision by Huisman and Kort (2015), we conclude that the increase of uncertainty, that is, a larger value of σ, raises both X_W^* and K_W^*. It implies that the social optimal investment is delayed with a greater volatility, which leads to the adoption of a larger project. This result shows that volatility influences social planner's investment decision in the same way as it influences the firm's investment decision. Moreover, the increase in drift rate parameter, i.e., a larger value of μ, raises X_W^* and K_W^* as well. The implication is that the social planner delays and takes on a larger project upon investment when market grows faster. This is due to the fact that future market demand is taken into consideration when making investment decisions. A faster growing market yields a higher demand in the future. Thus, more capacity is needed to satisfy such demand. It delays investment because of the prolonged waiting for a larger market demand to be reached.

4.3.2 Subsidized Profit Maximization Investment

As mentioned above, we study monopoly firm's investment decision under subsidy regulation. More specifically, we get the insight of how subsidy influences firm's optimal investment decision, in order to come up with a subsidy that can achieve either the first-best or the second-best outcome. For the given linear demand function and subsidy flow scheme $s(X(t),K,\tilde{s})=\tilde{s}p(X(t),K)K$, the firm internalizes the subsidy remuneration into the decision making. Substitute the price in firm's objective function (4.3) with subsidized price $(1+\tilde{s})p(X(t),K)$. For a given capacity K and $X(0)=X$, the firm's value function is equal to

$$V(X,K,\tilde{s}) = (1+\tilde{s})\left[\frac{XK}{r-\mu} - \frac{(1-\alpha)X\eta K^2}{r-\mu} - \frac{\alpha\eta K^2}{r}\right] - \delta K. \tag{4.8}$$

From this value function of the monopoly firm, we can derive the firm's optimal investment decision and the following proposition.

Proposition 4.2. *Given subsidy flow $s(X(t),K,\tilde{s}) = \tilde{s}p(X(t),K)K$, the optimal investment threshold $X^*(\tilde{s})$ and investment capacity $K^*(\tilde{s})$ satisfy the equations*

$$X(K) = \frac{\beta(r-\mu)}{(\beta-1)(1-(1-\alpha)\eta K)}\left(\frac{\alpha\eta K}{r} + \frac{\delta}{1+\tilde{s}}\right) \tag{4.9}$$

and

$$2\alpha(1-\alpha)\eta^2K^2 + \frac{r\delta}{1+\tilde{s}}(\beta+1)(1-\alpha)\eta K + \alpha(\beta-2)\eta K - \frac{r\delta}{1+\tilde{s}} = 0. \tag{4.10}$$

First note that when there is no subsidy, e.g., $\tilde{s} = 0$, we get the monopoly investment decision $(X^*(0),K^*(0))$. By comparing with the social optimal investment decision (X_W^*,K_W^*), it holds that $X^*(0) = X_W^*$ and $K^*(0) = K_W^*/2$. This indicates that when there is no subsidy under linear demand, the firm and the welfare maximizer have the same investment threshold, but the social optimal investment capacity is twice of the firm's capacity. This result is consistent with the finding by Huisman and Kort (2015), where linear demand is considered as well. We then study the influence of subsidy on firm's optimal investment, which is summarized in the following corollary.

Corollary 4.1. *Subsidy flow, $s(X(t),K,\tilde{s}) = \tilde{s}p(X(t),K)K$, makes the firm invest earlier and less. Unconditional subsidy cannot align firm's investment decision to the social optimal investment decision.*

Subsidy motivates the firm to invest earlier because monetary transfer increases the firm's expected value, which provides an incentive for the firm to enter the market earlier, when market demand is smaller. This leads to a smaller capacity being invested under subsidy regulation. Because firm's investment capacity without subsidy is already only half of the social optimal capacity, subsidy regulations makes firm invest less than half of the social optimal capacity, and thus deviate from the social optimal decision. So the unconditional subsidy cannot align profit and welfare maximizer's investment decisions. Another insight is that it is difficult to align two decision variables with just one subsidy rate parameter $\tilde{s}$ in unconditional subsidy regulation. This is because both decision variables, $X^*(\tilde{s})$ and $K^*(\tilde{s})$, are changing with $\tilde{s}$. Intuitively, two parameters and a more complicated subsidy regulation scheme will be needed. Next, we check another subsidy flow with one parameter as well, $s(X(t),K,\tilde{s}) = \tilde{s}K$ and $\tilde{s} < r\delta$. This unconditional subsidy regulation makes the firm invest in the way as described by the following proposition. The firm's value function $V(X,K,\tilde{s})$ and the proof of the proposition can be found in the appendix.

Proposition 4.3. *Subsidy flow $s(X(t),K,\tilde{s}) = \tilde{s}K$ makes the firm invest at threshold $X^*(\tilde{s})$ with capacity $K^*(\tilde{s})$. $X^*(\tilde{s})$ and $K^*(\tilde{s})$ satisfy*

$$X(K) = \frac{\beta(r-\mu)}{(\beta-1)(1-\eta K(1-\alpha))}\left(\frac{\alpha\eta K}{r} - \frac{\tilde{s}}{r} + \delta\right)$$

and

$$2\alpha(1-\alpha)\eta^2K^2 + (\beta+1)(1-\alpha)(r\delta-\tilde{s})\eta K + \alpha(\beta-2)\eta K - (r\delta-\tilde{s}) = 0.$$

Similar to the subsidy flow $\tilde{s}p(X(t),K)K$, we have $\mathrm{d}K^*(\tilde{s})/\mathrm{d}\tilde{s} < 0$ and $\mathrm{d}X^*(\tilde{s})/\mathrm{d}\tilde{s} < 0$, implying subsidy flow $s(X(t),K,\tilde{s}) = \tilde{s}K$ influences firm's investment decision in the same way and cannot achieve the first best if it is unconditional, i.e., implemented at Brownian motion level $X(0) < X^*$. For the lump sum subsidy transfer, it works the same as $s(X(t),K,\tilde{s}) = \tilde{s}K$ as long as its lump sum subsidy rate is $\tilde{s}/(r\delta)$.

4.3.3 Second-best outcome for unconditional subsidy

Because unconditional subsidy does not yield the first-best outcome, we want to find out the second-best outcome that can be achieved by subsidy regulation. Given the firm has invested at threshold $X^*(\tilde{s})$ with capacity $K^*(\tilde{s})$, the expected social surplus is equal to

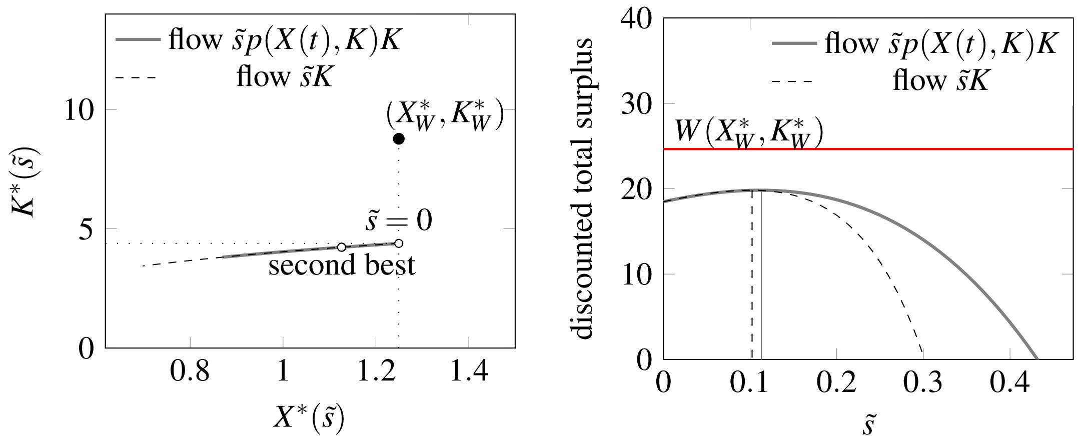
$$W(\tilde{s}) = K^*(\tilde{s})\left(\frac{X^*(\tilde{s})}{r-\mu} - \frac{(1-\alpha)X^*(\tilde{s})\eta K^*(\tilde{s})}{2(r-\mu)} - \frac{\alpha\eta K^*(\tilde{s})}{2r} - \delta\right).$$

Because $X^*(\tilde{s}) < X^*(0)$ for $\tilde{s} > 0$, $W(\tilde{s})$ needs to be compared at a predetermined point in time such as $X^*(0)$ with a stochastic discount factor $(X^*(0)/X^*(\tilde{s}))^{\beta}$.The optimal subsidy rate $\tilde{s}^*$ that yields the second-best outcome satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}\left(\frac{X^*(0)}{X^*(\tilde{s})}\right)^{\beta} W(\tilde{s})\bigg|_{\tilde{s}=\tilde{s}^*} = \left(\frac{X^*(0)}{X^*(\tilde{s}^*)}\right)^{\beta}\left(-\frac{\beta W(\tilde{s}^*)}{X^*(\tilde{s}^*)}\,\frac{\mathrm{d}X^*(\tilde{s})}{\mathrm{d}s}\bigg|_{\tilde{s}=\tilde{s}^*} + \frac{\mathrm{d}W(\tilde{s})}{\mathrm{d}s}\bigg|_{\tilde{s}=\tilde{s}^*}\right) = 0, \tag{4.11}$$

or $\tilde{s}^* = 0$ if $\left(\frac{X^*(0)}{X^*(\tilde{s})}\right)^{\beta} W(\tilde{s})$ decreases[1] with $\tilde{s}$. $\tilde{s}^* = 0$ implies that the second-best outcome for unconditional subsidy is to implement no subsidy.

[1]Note that we can rule out the situation where $\frac{\mathrm{d}}{\mathrm{d}\tilde{s}}\left(\frac{X^*(0)}{X^*(\tilde{s})}\right)^{\beta} W(\tilde{s}) > 0$. This implies an infinite amount of monetary transfer to the firm.



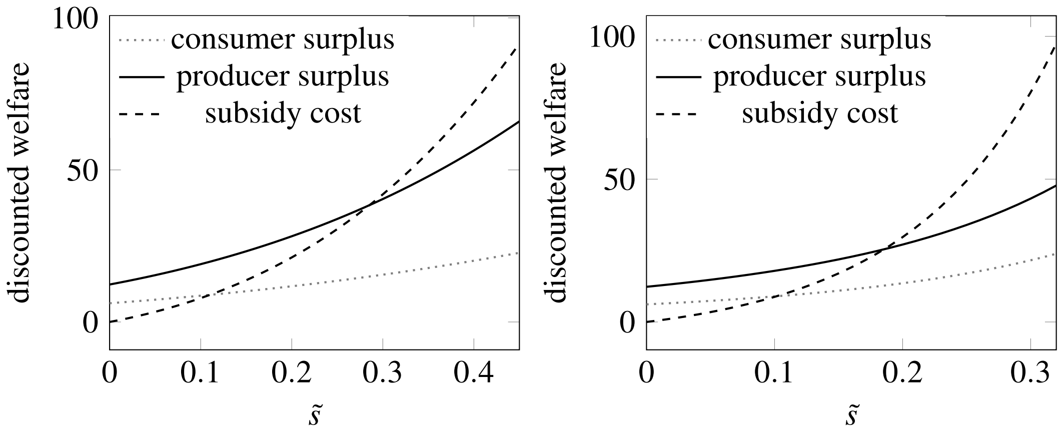
(a) Illustration of $(X^*(\tilde{s}), K^*(\tilde{s}))$. (b) Illustration of $W(\tilde{s}) = W(X^*(\tilde{s}), K^*(\tilde{s}))$.

Figure 4.1: Illustration of $(X^*(\tilde{s}), K^*(\tilde{s}))$, and $W(\tilde{s})$. Parameter values are $\alpha = 0.5$, $\mu = 0.02$, $r = 0.1$, $\sigma = 0.01$, $\eta = 0.05$, $\delta = 10$.

Figure 4.1 demonstrates the first-best, second-best outcome and the firm's optimal investment threshold $X^*(\tilde{s})$ and investment capacity $K^*(\tilde{s})$. It is shown that without subsidy, i.e., $\tilde{s} = 0$, the firm invests at the social optimal threshold $X^*(0) = X_W^*$ with half of the social optimal capacity $K^*(0) = K_W^*/2$. The half capacity result can be derived by comparing solutions for quadratic equations (4.7) and (4.10). For the given parameter values, $K^*(0) = 4.385$ and $K_W^* = 8.770$. This is consistent with the findings by Huisman and Kort (2015). Figure 4.1a also shows that as $\tilde{s}$ increases, the firm's optimal investment decision $(X^*(\tilde{s}), K^*(\tilde{s}))$ deviates further from the first-best outcome. Moreover, the two subsidy flows $\tilde{s}p(X(t),K)K$ and $\tilde{s}K$ have similar influence on firm's investment decision. We can see this from the overlap of the two curves[2] for $(X^*(\tilde{s}), K^*(\tilde{s}))$. Besides, the two subsidy flows have the same second-best outcome $(X^*(\tilde{s}^*), K^*(\tilde{s}^*))$ as shown in Figure 4.1a. This is due to the fact that $W(\tilde{s})$ has the same expression for the two unconditional subsidy flows. However, the second-best outcome is generated by different subsidy rates as illustrated by Figure 4.1b. The subsidy rate is $\tilde{s}^* = 0.113$ for subsidy flow $\tilde{s}p(X(t),K)K$ and $\tilde{s}^* = 0.102$ for flow $\tilde{s}K$. Figure 4.1b also illustrates the discounted total surplus generated by the firm's optimal investment decision. Under unconditional subsidy, the social surplus generated by the firm's investment decision is always below the social optimal surplus, implying that unconditional subsidy cannot lead to social optimum. Figure 4.1b shows that as the subsidy rate goes up, the social surplus first increases and then decreases. Because, the social surplus $W(\tilde{s})$ consists of producer surplus, consumer surplus and subsidy costs, we further analyze this by the illustration of Figure 4.2.

Figure 4.2 demonstrates the discounted consumer surplus $\left(\frac{X(0)}{X^*(\tilde{s})}\right)^\beta CS(X^*(\tilde{s}), K^*(\tilde{s}))$, the discounted producer surplus $\left(\frac{X(0)}{X^*(\tilde{s})}\right)^\beta V(X^*(\tilde{s}), K^*(\tilde{s}), \tilde{s})$ and the discounted subsidy

[2]We didn't plot $(X^*(\tilde{s}), K^*(\tilde{s}))$ that generates negative $W^*(\tilde{s})$.



(a) Subsidy flow $\tilde{s}p(X(t),K)K$. (b) Subsidy flow $\tilde{s}K$.

Figure 4.2: Illustration of $CS(\tilde{s})$, $PS(\tilde{s})$ and $C(\tilde{s})$. Parameter values are $\alpha = 0.5$, $\mu = 0.02$, $r = 0.1$, $\sigma = 0.01$, $\eta = 0.05$, $\delta = 10$.

cost $\left(\frac{X(0)}{X^*(\tilde{s})}\right)^{\beta} C(\tilde{s})$ to a predetermined time $X^*(0)$. Note that for subsidy flow $\tilde{s}p(X(t),K)K$, the expected subsidy cost is equal to

$$C(\tilde{s}) = \tilde{s}K^*(\tilde{s})\left(\frac{X^*(\tilde{s})}{r-\mu} - \frac{(1-\alpha)X^*(\tilde{s})\eta K^*(\tilde{s})}{r-\mu} - \frac{\alpha\eta K^*(\tilde{s})}{r}\right).$$

For subsidy flow $\tilde{s}K$, the expected subsidy cost is $\tilde{s}K^*(\tilde{s})/r$. Figure 4.2 shows that the discounted consumer surplus, producer surplus and subsidy costs increase with unconditional subsidy rate $\tilde{s}$, despite the fact that $\tilde{s} > 0$ makes the firm invest earlier and less. It is intuitive that the discounted producer surplus increases with the subsidy rate. This is because for two subsidy rates $\tilde{s}_1 > \tilde{s}_2 \geq 0$, the firm can always choose investment decision $(X^*(\tilde{s}_2), K^*(\tilde{s}_2))$ for subsidy rate $\tilde{s}_1$, which would yield a producer surplus that is equal to $PS(X^*(\tilde{s}_2), K^*(\tilde{s}_2)) + \tilde{s}_1 p(X^*(\tilde{s}_2), K^*(\tilde{s}_2))K^*(\tilde{s}_2)$. The fact that the firm chooses investment decision $(X^*(\tilde{s}_1), K^*(\tilde{s}_1))$ implies it generates larger producer surplus. The consumer surplus also increases with $\tilde{s}$ because though the firm's output decreases with a larger $\tilde{s}$, the firm starts production earlier. So the consumption also starts earlier, which is preferred by the consumers and yields a larger consumer surplus. The discounted subsidy cost also increases with $\tilde{s}$. This is intuitive because otherwise the government should provide an infinite subsidy rate given that both producer and consumer surpluses increase with $\tilde{s}$. When $\tilde{s}$ is small, the subsidy cost grows slower than the sum of producer and consumer surplus. This is illustrated by an increasing total surplus in Figure 4.1b. As $\tilde{s}$ increases, the subsidy cost increases faster than the sum of consumer and producer surplus. This leads to the decrease of total surplus in Figure 4.1b.

4.3.4 Optimal conditional subsidy

Though unconditional subsidy does not yield the first-best outcome, it is still possible to align firm's optimal investment decision to the social optimal decision through a conditional subsidy, that is, the subsidy implemented at a specific Brownian motion level. This is due to the same investment threshold of the firm and social planner without subsidy regulation, $X_W^* = X^*(0)$. We take that as one decision variable already being aligned. Then it is only necessary to align the investment capacities by choosing the subsidy rate parameter. The optimal conditional subsidy regulation is given by the following proposition.

Proposition 4.4. *The optimal conditional subsidy is to introduce subsidy at the social optimal investment threshold X_W^* with the following subsidy rate:*

$$\tilde{s}^* = \begin{cases} \frac{X_W^* - (r-\mu)\delta}{2(r-\mu)\delta - X_W^*} & \textit{for subsidy flow } \tilde{s}p(X(t),K)K, \\ \frac{rX_W^* - r(r-\mu)\delta}{r-\mu} & \textit{for subsidy flow } \tilde{s}K, \\ \frac{X_W^* - (r-\mu)\delta}{\delta(r-\mu)} & \textit{for lump sum subsidy } \tilde{s}K. \end{cases}$$

The optimal conditional subsidy described in this proposition aligns firm's and social planner's investment decision, and generates the first-best outcome. Because there is no asymmetry of information on investment costs, according to Broer and Zwart (2013), a conditional subsidy described in Proposition 4.4 can be interpreted simply as the regulator tells the monopolist when to invest and how much to invest. The changes in market parameters influence the dynamic optimal subsidy rates, and the influence is summarized in the following corollary.

Corollary 4.2. *When the volatility rate σ or drift parameter μ increases, the firm needs to be subsidized more in order to invest at the social optimal capacity level.*

This is because an increase in σ or μ makes the social planner invest later and more. In order for the firm to catch up with the social optimal capacity, either to prepare for positive future demand shocks because of larger σ or to satisfy a larger anticipated future market demand growth because of larger market trend μ, more monetary support needs to be transferred to the firm.

4.4 Non-linear Demand

From previous section, it is now clear that with linear inverse demand function, unconditional subsidy support does not align firm and social planner's investment decision. A possible reason might be the linear demand shocks. In this section, we study nonlinear demand shocks and check the performance of the same subsidy regulations in previous section. Suppose

$$p(t) = X(t)K(t)^{-\gamma}$$

with $0 < \gamma < 1$, and $X(t)$ follows geometric Brownian motion of (4.1). Investment costs are of the form[3] $\delta_0 + \delta_1 K(t)$ with $\delta_0 \geq 0$ and $\delta_1 > 0$. In the following analysis, we first discuss the first-best outcome and then check whether unconditional subsidy makes the monopolist deviate or converge to the social optimal investment. Later we focus on the optimal subsidy regulation policy.

4.4.1 First-best benchmark

The producer surplus equals to the value of investment, i.e., expected discounted profit flows after investment minus investment costs. For a given investment capacity K and geometric Brownian motion level $X(0) = X$, the producer surplus at X is given by

$$PS(X,K) = V(X,K) = \frac{XK^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 K.$$

For a given level of K, the producer surplus increases with X. The reason is that for a given output K, a larger X implies larger market demand and higher market prices, which makes the firm more profitable and thus generates larger producer surplus. Given the firm produces an output of K, the discounted instantaneous consumer surplus at $X(t)$ is

$$cs(X(t),K) = \int_{p}^{\infty} \left(\frac{X(t)}{p}\right)^{\frac{1}{\gamma}} dp = \frac{\gamma}{1-\gamma} X^{\frac{1}{\gamma}}(t) p^{\frac{\gamma-1}{\gamma}} \Big|_{X(t)K^{-\gamma}}^{\infty} = \frac{\gamma}{1-\gamma} X(t) K^{1-\gamma}.$$

At $X(0) = X$, the expected consumer surplus is

$$CS(X,K) = \frac{\gamma}{1-\gamma} \frac{XK^{1-\gamma}}{r-\mu}.$$

The insight for consumer surplus is the same as that under linear demand. For a given X, the consumer surplus increases with K because more output decreases market prices. For a given K, the consumer surplus increases with X because consumer's willingness to pay increases. The expected social surplus is the sum of producer and consumer surplus and is

[3]We take a different cost structure than the linear demand because of two reasons. First reason is that the cost structure δK does not yield any solution for firm's investment decision under non-linear demand. Second reason is that the cost structure $\delta_0 + \delta_1 K$ does not change the main results obtained under linear demand.

given by

$$W(X,K) = \frac{1}{1-\gamma}\frac{XK^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 K.$$

From the social welfare function, we can derive the social optimal investment decision as the first-best benchmark. It is summarized in the following proposition.

Proposition 4.5. *The social optimal investment threshold is*

$$X_W^* = (r-\mu)\delta_1 \left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}$$

and the social optimal investment capacity is

$$K_W^* \equiv K_W^*(X_W^*) = \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}.$$

A further analysis on the influence of market volatility yields similar insight as that under linear market demand structure. Because $\partial\beta/\partial\sigma < 0$, $\partial X_W^*/\partial\beta < 0$, and $\partial K_W^*/\partial\beta < 0$, it can be concluded that $\partial X_W^*/\partial\sigma > 0$ and $\partial K_W^*/\partial\sigma > 0$, implying a non-linear demand structure like the iso-elastic demand does not change the standard real option result that a greater volatility delays investment and leads to installing a larger project.

4.4.2 Subsidized Profit Maximization Investment

In this subsection, subsidy flows $s(X(t),K,\tilde{s}) = \tilde{s}p(X(t),K)K$ and $s(X(t),K,\tilde{s}) = \tilde{s}K$ are considered. The lump sum subsidy transfer $s(K,\tilde{s}) = \tilde{s}(\delta_0 + \delta_1 K)$ is analyzed in more detail than that under the linear demand structure because of the fixed cost, δ_0, from investment. This makes it behave a little differently from the subsidy flow $\tilde{s}K$. The focus of the analysis is on how subsidy influences firm's investment decision. Moreover, it compares the influence of subsidy under non-linear demand with the influence under linear demand.

Proposition 4.6. *When subsidy flow is $s(X(t),K,\tilde{s}) = \tilde{s}p(X(t),K)K$, firm's optimal investment threshold $X^*(\tilde{s})$ and investment capacity $K^*(\tilde{s})$ are equal to*

$$\begin{aligned} X^*(\tilde{s}) &= \frac{\delta_1(r-\mu)}{(1-\gamma)(1+\tilde{s})}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}, \\ K^*(\tilde{s}) &= \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}. \end{aligned}$$

When subsidy flow is $s(X(t),K,\tilde{s}) = \tilde{s}K$ and $\tilde{s} < r\delta_1$, firm's optimal investment threshold

$X^*(\tilde{s})$ and investment capacity $K^*(\tilde{s})$ are given by

$$
\begin{aligned}
X^*(\tilde{s}) &= \frac{r-\mu}{1-\gamma}\left(\delta_1 - \frac{\tilde{s}}{r}\right)\left(\frac{\delta_0\beta(1-\gamma)}{(\delta_1-\tilde{s}/r)(\beta\gamma-1)}\right)^{\gamma}, \\
K^*(\tilde{s}) &= \frac{\delta_0\beta(1-\gamma)}{(\delta_1-\tilde{s}/r)(\beta\gamma-1)}.
\end{aligned}
$$

When the lump sum subsidy transfer is $s(K,\tilde{s}) = \tilde{s}(\delta_0+\delta_1 K)$, firm's optimal investment threshold $X^*(\tilde{s})$ and investment capacity $K^*(\tilde{s})$ are equal to

$$
\begin{aligned}
X^*(\tilde{s}) &= \frac{\delta_1(r-\mu)(1-\tilde{s})}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}, \\
K^*(\tilde{s}) &= \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}.
\end{aligned}
$$

By comparing (X_W^*, K_W^*) with $(X^*(0), K^*(0))$, we find that without subsidy regulation, the firm invests later than the social planner but with the social optimal capacity. This is different from the linear demand structure, where the firm invests at the same time as the social planner but with half of the social optimal capacity when $\tilde{s} = 0$. For unconditional subsidy, Proposition 4.6 shows that subsidy makes the firm invest earlier, the same as the linear demand structure. Another insight of the three subsidy regulations is that subsidy flow $\tilde{s}K$ influences the firm's optimal investment capacity, but subsidy flow $\tilde{s}p(X(t),K)K$ and lump sum subsidy $\tilde{s}(\delta_0+\delta_1 K)$ do not. This is different from that under linear demand structure, where all the three subsidy regulations make firm invest less.

For the unconditional subsidy flow $\tilde{s}p(X(t),K)K$ and lump sum subsidy $\tilde{s}(\delta_0+\delta_1 K)$, because $X^*(0) > X_W^*$ and subsidy regulation makes firm invest earlier than trigger $X^*(0)$, it is possible to align firm's and social optimal investment threshold by choosing appropriate subsidy rate $\tilde{s}$. This implies that unconditional subsidy can reach the first-best outcome for subsidy flow $\tilde{s}p(X(t),K)K$ and lump sum subsidy $\tilde{s}(\delta_0+\delta_1 K)$. Whereas for subsidy flow $\tilde{s}K$, unconditional subsidy not only makes the firm invest earlier but also makes the firm invest with a capacity that is larger than the social optimal capacity. This implies that the first-best outcome cannot be reached for unconditional subsidy flow $\tilde{s}K$. But a conditional subsidy flow $\tilde{s}K$ can achieve the first-best outcome. This is because subsidy motivates the firm to invest earlier than $X^*(0)$, a conditional subsidy can be implemented such that the firm invests at X_W^* with K_W^* for subsidy flow $\tilde{s}K$. In fact, the optimal conditional subsidy can be implemented for all the three, the same as under the linear demand structure. We summarize the optimal unconditional and conditional subsidy regulations in the following proposition.

Proposition 4.7. *Unconditional subsidy regulation implemented at $X(0)$ is optimal for subsidy flow $\tilde{s}p(X(t),K)K$ and lump sum subsidy transfer $\tilde{s}(\delta_0+\delta_1 K)$ if the subsidy rate*

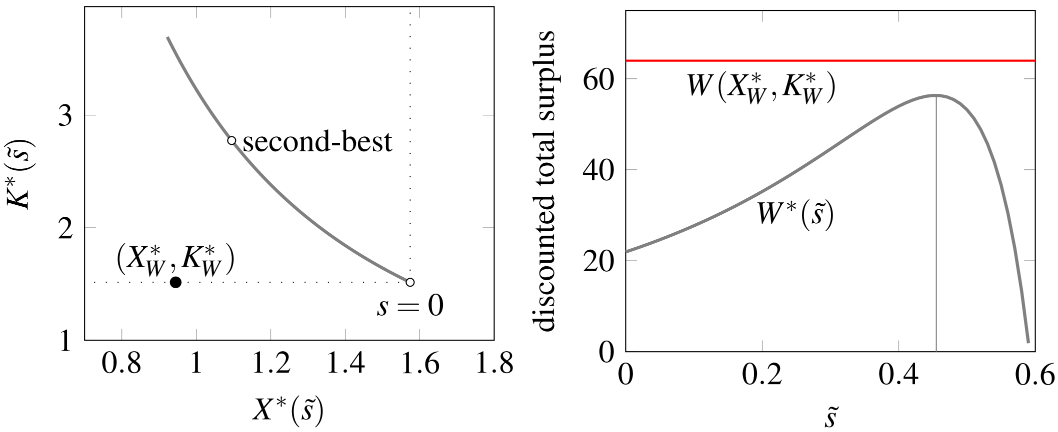
$\tilde{s}^*$ is equal to,

$$\tilde{s}^* = \begin{cases} \frac{\gamma}{1-\gamma} & \text{for subsidy flow } \tilde{s}p(X(t),K)K, \\ \gamma & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

Conditional subsidy regulation implemented at X_W^* is optimal if the subsidy rate $\tilde{s}^*$ is given by

$$\tilde{s}^* = \begin{cases} \frac{\gamma}{1-\gamma} & \text{for subsidy flow } \tilde{s}p(X(t),K)K, \\ r\gamma\delta_1 & \text{for subsidy flow } \tilde{s}K, \\ \gamma & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

With the optimal subsidy regulation, the firm's investment decision is aligned to the social optimal decision and leads to the first-best outcome. This result is the same as under the linear demand structure. Recall from the previous section that for unconditional subsidy, the second-best outcome is to implement no subsidy at all. In the following analysis, we check whether this is also true for iso-elastic demand structure. Note that unconditional subsidy flow $\tilde{s}p(X(t),K)K$ and lump sum subsidy $\tilde{s}(\delta_0+\delta_1 K)$ can achieve the first-best outcome. So our focus is on the unconditional subsidy flow $\tilde{s}K$.



(a) Illustration of $(X^*(\tilde{s}),K^*(\tilde{s}))$. (b) Illustration of $W(\tilde{s}) = W(X^*(\tilde{s}),K^*(\tilde{s}))$.

Figure 4.3: Illustration of $(X^*(\tilde{s}),K^*(\tilde{s}))$, and $W^*(\tilde{s})$ for subsidy flow $\tilde{s}K$. Parameter values are $\mu = 0.02$, $r = 0.1$, $\sigma = 0.01$, $\gamma = 0.5$, $\delta_0 = 2$, $\delta_1 = 10$.

Figure 4.3a demonstrates the firm's optimal investment capacity $K^*(\tilde{s})$ and optimal investment threshold $X^*(\tilde{s})$ as functions of subsidy rate $\tilde{s}$. It is clear that when $\tilde{s} = 0$, $K^*(0) = K_W^*$. As $\tilde{s}$ increases, $K^*(\tilde{s})$ deviates from social optimal K_W^*, but $X^*(\tilde{s})$ is getting close to X_W^*. Figure 4.3b shows the total surplus, discounted to a predetermined time $X^*(0)$, as a function of unconditional subsidy rate. As illustrated, there exists a subsidy rate that generates the highest level of social welfare for unconditional subsidy. The subsidy rate that generates the second-best outcome is $\tilde{s} = 0.455$. As shown in Figure 4.3b, the total surplus for the second-best outcome is below the social optimal welfare that is also discounted to $X^*(0)$. This result is similar to that under linear demand structure for the unconditional subsidy regulation.

4.5 Conclusion

This chapter analyzes investment decision of a profit maximizer under subsidy regulation and how to align this decision to social optimal decision through optimal subsidy. We show that unconditional subsidy introduced from the beginning accelerates the investment of a monopoly firm. Under linear demand structure, unconditional subsidy regulation cannot align the profit and welfare maximizers' investment decisions. Moreover, it decreases monopolist's optimal investment capacity and results in smaller social surplus. There is conditional subsidy regulation that aligns the firm's investment decision to the social optimal decision. This optimal conditional subsidy requires to introduce subsidy at the social optimal investment threshold. For non-linear iso-elastic demand, depending on the form of subsidy regulations, it is possible to implement unconditional subsidy to align profit maximizing and social optimal investment decisions. The conditional subsidy can also be implemented in a similar way as under linear demand structure. If we dismiss the efficiency loss when collecting and allocating the taxation, the aligned profit maximizer's investment decision can lead to the first-best outcome for both the linear and non-linear market demand.

4.6 Appendix

This appendix contains proofs of the propositions for the linear and iso-elastic demand functions.

Proof of Proposition 4.1 The social optimal investment capacity $K_W(X)$ maximizes $W(X,K)$ and is equal to

$$K_W(X) = \frac{rX - r(r-\mu)\delta}{\eta[(r-\mu)\alpha + r(1-\alpha)X]}, \tag{4.12}$$

which is equivalent to

$$X = \frac{(r-\mu)(r\delta + \alpha\eta K_W(X))}{r(1-(1-\alpha)\eta K_W(X))}.$$

Let the option value before social planner's investment be $A_W X^\beta$. The value matching and smooth pasting at the social optimal investment threshold X_W^* yield

$$W(X_W^*, K_W(X_W^*)) \quad = \quad A_W X_W^{*\beta},$$

$$\left.\frac{\partial W(X_W^*, K_W(X_W^*))}{\partial X}\right|_{X=X_W^*} = \beta A_W X_W^{*\beta-1}.$$

Then we have the following equation

$$\frac{X_W^*}{r-\mu}\frac{\beta-1}{\beta}(2-(1-\alpha)\eta K_W(X_W^*)) - \frac{\alpha\eta K_W(X_W^*)}{r} - 2\delta = 0. \tag{4.13}$$

Combining (4.12) and (4.6), we get the social optimal investment capacity K_W^* satisfies the following implicit equation

$$\alpha(1-\alpha)\eta^2 K^2 + r\delta(\beta+1)(1-\alpha)\eta K + \alpha(\beta-2)\eta K - 2r\delta = 0. \tag{4.14}$$

Proof of Proposition 4.2 For $X(0) = X$, the optimal investment capacity $K(X,\tilde{s})$ maximizes the investment value and thus satisfies the following first order condition

$$\frac{X}{r-\mu} - \frac{\delta}{1+\tilde{s}} = \frac{2\alpha\eta K}{r} + \frac{2(1-\alpha)X\eta K}{r-\mu}.$$

Thus,

$$K(X,\tilde{s}) = \frac{r(1+\tilde{s})X - r(r-\mu)\delta}{2\eta(1+\tilde{s})[(r-\mu)\alpha + r(1-\alpha)X]}. \tag{4.15}$$

Let the option value before investment be AX^β, $\beta > 2$ from assumptions in the model of additive demand function. From the value matching and smooth pasting at the optimal investment threshold X^*, then

$$\begin{aligned} V(X^*, K(X^*,\tilde{s}), \tilde{s}) &= AX^{*\beta}, \\ \left.\frac{\partial V(X, K(X,\tilde{s}), \tilde{s})}{\partial X}\right|_{X=X^*} &= \beta A X^{*\beta-1}. \end{aligned}$$

This yields that $X(\tilde{s})$ satisfies the following equation

$$\frac{X^*}{r-\mu}\frac{\beta-1}{\beta}(1-(1-\alpha)\eta K(X^*)) - \frac{\alpha\eta K(X^*)}{r} - \frac{\delta}{1+\tilde{s}} = 0. \tag{4.16}$$

Solving (4.15) and (4.16) yields that the optimal investment capacity $K^*(\tilde{s})$ satisfies the quadratic form

$$2\alpha(1-\alpha)\eta^2K^2+\frac{r\delta}{1+\tilde{s}}(\beta+1)(1-\alpha)\eta K+\alpha(\beta-2)\eta K-\frac{r\delta}{1+\tilde{s}}=0. \tag{4.17}$$

Proof of Corollary 4.1 Denote $\delta/(1+\tilde{s})=x$, then from (4.10), it can be derived that

$$\frac{\mathrm{d}K^*}{\mathrm{d}x}\left(4\alpha(1-\alpha)\eta^2K^*+rx(\beta+1)(1-\alpha)\eta+\alpha(\beta-2)\right)=r\left(1-(\beta+1)(1-\alpha)\eta K^*\right).$$

This implies that $\mathrm{d}K^*/\mathrm{d}x>0$, i.e., $\mathrm{d}K^*(\tilde{s})/\mathrm{d}\tilde{s}<0$. So the subsidy in the market motivates the firm to invest less. Moreover, from (4.15), the profit maximizer's optimal investment threshold is also influenced by the subsidy. It holds that

$$\frac{\mathrm{d}X^*}{\mathrm{d}x}\frac{1-2(1-\alpha)\eta K^*}{r-\mu}=\frac{\mathrm{d}K^*}{\mathrm{d}x}\left(\frac{2(1-\alpha)\eta X^*}{r-\mu}+\frac{2\alpha\eta}{r}+1\right).$$

This yields that $\mathrm{d}X^*/\mathrm{d}x>0$, i.e., $\mathrm{d}X^*(\tilde{s})/\mathrm{d}\tilde{s}<0$. The subsidy also makes the profit maximizer invest earlier.

Proof of Proposition 4.3 For a given capacity K and $X(0)=X$, the value for the expected discounted profit flow is

$$V(X,K,\tilde{s})=\frac{XK}{r-\mu}-\eta K^2\left(\frac{\alpha}{r}+\frac{(1-\alpha)X}{r-\mu}\right)+\frac{\tilde{s}K}{r}-\delta K.$$

The optimal capacity for a given X and $\tilde{s}$ maximizes the value of the firm and is given by

$$K(X,\tilde{s})=\frac{rX+(r-\mu)\tilde{s}-r(r-\mu)\delta}{2\eta\left[(r-\mu)\alpha+r(1-\alpha)X\right]}.$$

For a given capacity size K, by value matching and smooth pasting at the investment threshold, it can be derived that

$$X(K,\tilde{s})=\frac{\beta(r-\mu)}{\beta-1}\frac{\alpha\eta K-\tilde{s}+r\delta}{r[1-\eta K(1-\alpha)]}.$$

Combining $K(X,\tilde{s})$ and $X(K,\tilde{s})$, we get that the optimal investment capacity $K^*(\tilde{s})$ satisfies the implicit expression

$$2\alpha(1-\alpha)\eta^2K^2+(\beta+1)(1-\alpha)(r\delta-\tilde{s})\eta K+\alpha(\beta-2)\eta K-(r\delta-\tilde{s})=0.$$

Proof of Proposition 4.4 For subsidy flow $\tilde{s}p(X(t),K)K$ and $\tilde{s}K$, the optimal subsidy rate can be derived by letting $K_W(X_W)=K(X_W,\tilde{s})$. For lump sum subsidy, the optimal subsidy rate is equal to $\tilde{s}^*/(r\delta)$ given $\tilde{s}^*$ as the optimal subsidy rate for the flow $\tilde{s}K$.
Proof of Corollary 4.2 Larger σ leads to larger $\tilde{s}^*$ because of $\partial X_W^*/\partial\sigma>0$ and $\partial\tilde{s}^*/\partial X_W^*>0$. Thus, it holds that the optimal conditional subsidy rate $\tilde{s}^*$ increases with σ. Next, we check the influence of μ on $\tilde{s}^*$. Recall from previous analysis that $\partial X_W^*/\partial\mu>0$. Then for the three optimal conditional subsidy rates, we can get the following first order partial derivatives of $\tilde{s}^*$ with respect to μ

$$\frac{\partial\tilde{s}^*}{\partial\mu}=\begin{cases}\frac{\delta}{[2(r-\mu)\delta-X_W^*]^2}\left(X_W^*+(r-\mu)\frac{\partial X_W^*}{\partial\mu}\right) & \text{for flow } \tilde{s}p(X(t),K)K,\\ \frac{r}{(r-\mu)^2}\left(X_W^*+(r-\mu)\frac{\partial X_W^*}{\partial\mu}\right) & \text{for flow } \tilde{s}K,\\ \frac{1}{\delta(r-\mu)^2}\left(X_W^*+(r-\mu)\frac{\partial X_W^*}{\partial\mu}\right) & \text{for lump sum } \tilde{s}K.\end{cases}$$

It can be concluded that $\partial\tilde{s}^*/\partial\mu>0$.
Proof of Proposition 4.5 For given X, the investment capacity that maximizes social welfare is equal to

$$K_W(X)=\left(\frac{X}{(r-\mu)\delta_1}\right)^{1/\gamma}.$$

Let the social planner's value before investment be AX^β, then according to value matching and smooth pasting conditions at optimal investment threshold X_W^*, we get

$$\begin{aligned}AX_W^{*\beta} &= \frac{1}{1-\gamma}\frac{X_W^*K_W(X_W^*)^{1-\gamma}}{r-\mu}-\delta_0-\delta_1K_W(X_W^*),\\ \beta AX_W^{*\beta-1} &= \frac{1}{1-\gamma}\frac{K_W(X_W^*)^{1-\gamma}}{r-\mu}.\end{aligned}$$

This yields

$$X_W^*=(r-\mu)\delta_1\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^\gamma.$$

The corresponding investment capacity is given by

$$K_W^*\equiv K_W(X_W^*)=\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}.$$

To get the insight of how σ influences X_W^* and K_W^*, we can derive the following first order partial derivatives:

$$\begin{aligned}
\frac{\partial X_W^*}{\partial \beta} &= -\frac{\delta_0\gamma(1-\gamma)(r-\mu)}{(\beta\gamma-1)^2}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma-1} < 0,\\
\frac{\partial K_W^*}{\partial \beta} &= -\frac{\delta_0(1-\gamma)}{\delta_1(\beta\gamma-1)^2} < 0.
\end{aligned}$$

Proof of Proposition 4.6 For a given capacity size K and $X(0) = X$, the value of the expected discounted profit flow at X is equal to

$$V(X,K) = \begin{cases} \frac{XK^{1-\gamma}(1+\tilde{s})}{r-\mu} - \delta_0 - \delta_1 K & \text{for subsidy flow } \tilde{s}p(X(t),K)K,\\ \frac{XK^{1-\gamma}}{r-\mu} + \frac{\tilde{s}K}{r} - \delta_0 - \delta_1 K & \text{for subsidy flow } \tilde{s}K,\\ \frac{XK^{1-\gamma}}{r-\mu} - (1-\tilde{s})(\delta_0 + \delta_1 K) & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

Maximizing $V(X,K)$ with respect to K yields that the optimal capacity for a given X is given by

$$K(X) = \begin{cases} \left(\frac{X(1-\gamma)(1+\tilde{s})}{\delta_1(r-\mu)}\right)^{1/\gamma} & \text{for subsidy flow } \tilde{s}p(X(t),K)K,\\ \left(\frac{X(1-\gamma)}{(r-\mu)(\delta_1-\tilde{s}/r)}\right)^{1/\gamma} & \text{for subsidy flow } \tilde{s}K,\\ \left(\frac{X(1-\gamma)}{(r-\mu)(1-\tilde{s})\delta_1}\right)^{1/\gamma} & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

Substituting $K(X)$ into $V(X,K)$ gives the expected value as a function of X, i.e., $V(X)$. Let the value before investment threshold X^* be AX^{β}. Then the value matching and smooth pasting conditions at X^* yield

$$X^*(\tilde{s}) = \begin{cases} \frac{\delta_1(r-\mu)}{(1-\gamma)(1+\tilde{s})}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma} & \text{for subsidy flow } \tilde{s}p(X(t),K)K,\\ \frac{r-\mu}{1-\gamma}(\delta_1-\tilde{s}/r)\left(\frac{\delta_0\beta(1-\gamma)}{(\delta_1-\tilde{s}/r)(\beta\gamma-1)}\right)^{\gamma} & \text{for subsidy flow } \tilde{s}K,\\ \frac{\delta_1(r-\mu)(1-\tilde{s})}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma} & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

From the optimal investment threshold $X^*(\tilde{s})$, we can get that the optimal investment capacity $K^*(\tilde{s})$ is equal to

$$K^*(\tilde{s}) \equiv K^*(X^*(\tilde{s})) = \begin{cases} \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)} & \text{for subsidy flow } \tilde{s}p(X(t),K)K, \\ \frac{\delta_0\beta(1-\gamma)}{(\delta_1-\tilde{s}/r)(\beta\gamma-1)} & \text{for subsidy flow } \tilde{s}K, \\ \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)} & \text{for lump sum subsidy } \tilde{s}(\delta_0+\delta_1 K). \end{cases}$$

Proof of Proposition 4.7 Given in the text.

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